What is topology?

Art of Mathematics
TPEI Summer 2024

1 Introduction

Topology is the field of mathematics which studies those aspects of an object's shape which are unchanged after bending, stretching or sliding it around, but not (necessarily) after breaking or cutting it. We call the former, allowed changes in shape *continuous deformations*. Topology is more basic than geometry, as geometry considers distances and angles which may be altered by continuous deformations.

One kind of object which receives considerable attention in topology and appears in a good deal of artwork is the *knot*. In this lecture we'll explore knots from the perspective of topology, and we'll view the traditional artwork of several cultures which utilizes knots and related patterns. We'll discuss

2 Defining knots

Suppose you have a loop of string which you can stretch out or tighten down however you like, and you place it flat on a table. Note that it cannot cross over itself, because the over-crossing would need to leave the plane of the table. There are many shapes and positions in which you could place the loop on the table in this way; however, given any two such placements, you could always move the loop from one into the other simply by sliding it around and possibly tightening or loosening it.

For this reason we say that there is *essentially one way* of tracing out a loop in a plane, i.e. in two dimensions. Now what if instead of staying on the table, you're allowed to form the loop in three-dimensional space from a length of the same string, twisting it around however you like before tying the two ends together? We'll suppose that it can float freely wherever you situate it—with no effect from gravity—so that the string never touches itself, even though in your field vision it will appear to be crossing itself. You probably won't be surprised to hear that this is what we as mathematicians call a *knot*—any way of forming a loop (which does not intersect itself) in three-dimensional space.

As mathematician-artists we'll be interested in two questions:

1. How can we represent in knots in (roughly) two dimensions, as when drawing on paper or carving the face of wood or stone? Are there methods to produce such representations?

2. Thinking back to the essentially unique way of forming a loop in two-dimensional space, how many ways there are of tracing out knots which are really different, and can't be moved around, elongated or constricted until they match, without breaking the loop? How can we tell two distinct knots apart?

3 Representing knots

To represent a knot in two-dimensions, we start by imagining the shadow it would cast when placed between a distant source of light and a flat surface. In your mind's eye, you'll see the shadow as a curve drawn upon the surface, intersecting itself a number of times at points we call *crossings*. To fully capture the knot, we need to introduce one additional bit of information to the crossings: which strand of the curve is passing *over* and correspondingly which is passing *under*. You may draw this over-under information however you like; typically one leaves a gap in the under strand through passes the over strand. We call the resulting drawing a *planar diagram* of a knot. Going forward we may refer to a planar diagram simply as a knot itself, even though it is technically just a representation.

You may already be able to see in your imagination that the exact same knot, floating in space in exactly the same position, can cast very different shadows depending on where the light source and surface are situated. As a good exercise ahead of section 4, try visualizing a knot in space—even something as simple as a circle bent around a little—and then try drawing two shadows it could cast which have a different number of crossings.

4 Telling knots apart

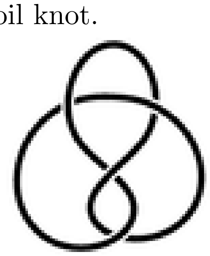
First, let's establish when we will consider two knots to be *essentially the same*. The idea is not unlike that of the loops in the introduction 1: If there's some way of getting two knots to match by sliding, bending, stretching or constricting one or both of them, then we consider them to be the same knot. In terms of planar diagrams, there is a tool for determining when a pair diagrams represent the same knot: There is a set of three transformations, called *Reidemeister moves*, which relate any two equivalent knots. In the late 1920s, mathematicians Kurt Reidemeister, James Alexander and Garland Briggs proved that, if knots K_1 and K_2 are equivalent, then there exists a sequence of Reidemeister moves which takes K_1 into K_2. The moves are shown below.

In general, verifying that two knots are equivalent is a very difficult task, but building upon this foundation, mathematicians seek efficient tools for distinguishing inequivalent knots. These are called *invariants*. An invariant is a property of or quantity associated with a knot which is unchanged under Reidemeister moves, and it follows that if an invariant differs for two knots, then they cannot be equivalent. Following are two examples which can be described in terms of planar diagrams:

1. **Chiral**—the knot is equivalent to its own mirror image.

(a) Left-handed trefoil knot.

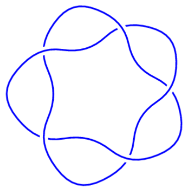
(b) Right-handed trefoil knot.



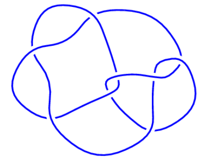
(c) Achiral figure-8 knot.

Figure 1: Knot chirality.

2. **Alternating**—as one traces along the knot, each over-crossing is followed by an under-crossing and vice versa (i.e. over- and under- alternate).



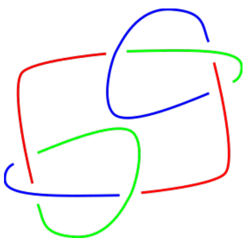
(a) Alternating knot: cinquefoil knot (Solomon's seal).



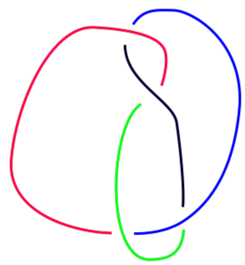
(b) Non-alternating 8-crossing knot.

Figure 2: Knot colorings.

3. **Tricolorability**—each of *strands* can be given one of 3 colors so that no adjacent strand has the same color. Here a strand is a segment of the knot which runs from one crossing to another and has no crossings in between.



(a) Tricolored granny knot.



(b) Non-tricolorable figure-eight knot.

Figure 3: Knot colorings.

5 Building knots

There are many methods and algorithms for generating knots. Here we describe a few mathematical constructions, which can help to rigorously describe the creations seen in art.

5.1 Braid closures

One method is to close off a braid by connecting corresponding endpoints on either side.

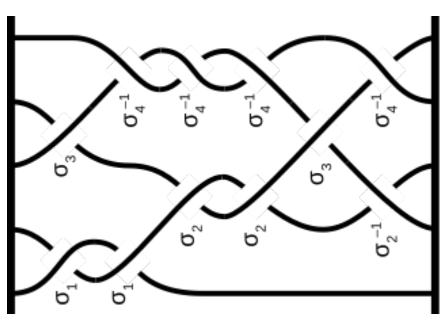


Figure 4: Braid.

Note that the result for an abitrary braid may not produce a *single* closed loop; their may be multiple loops—each a knot in its own right—which are intertwined. We call this intertwined collection of knots a *link*. **Alexander's theorem**, proved by James Alexander in 1923, states that every knot or link is the closure of a braid.

5.2 Skein relations

Another way to construct a knot is to start from a given knot–even just a circle, called the *unknot* by knot theorists—and to add, remove or alter crossings. The result could transform a knot into a link or vice versa. When two knots or links differ by such a change at a crossing, we say that a *skein relation* holds between them. There are two important species of skein relation:

5.2.1 Unoriented: Kauffman bracket

Replace one of the following three "patches" around a crossing in a knot or link with one of the others:

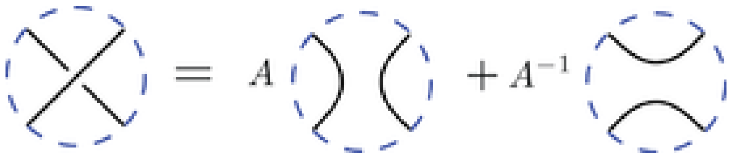


Figure 5: Kauffman bracket: unoriented skein relation.

A polynomial called the **Kauffman bracket** is defined in terms of this relation. This in turn is used to construct the **Jones polynomial** and **Khovanov homology** of a knot or link, two especially important algebraic knot invariants.

5.2.2 Oriented skein relation

This time we pick a direction in which to traverse the knot (or each individual loop in a link), which can be represented in a diagram with arrowheads. Then proceed as in the last section, now with the following three "paches":

Figure 6: Oriented skein relation.

This skein relation is closely related to two more important knot invariants: the **Alexander polynomial** and **(Heegaard) knot Floer homology**.

5.3 Mirror curves

It is speculated that something like the method of construction using mirror curves was employed by cultures across several continents to produce artistic knot representations, as seen in §6.1 below. For this we do the following:

1. Construct a rectangular grid with square cells.

2. Draw one or more loops by connecting the midpoints of the square cells, as if a light ray were bouncing off the outer boundary walls and passing straight through the inner ones.

3. At your choice of crossings, draw in a horizontal or vertical bar. This acts as a mirror on the light ray, and resolves the crossing as in the Kauffman bracket in §5.2.1 above.

4. Apply over-under information to each remaining (unresolved/unmirrored) crossing. The easiest way to do this is in an alternating fashion.

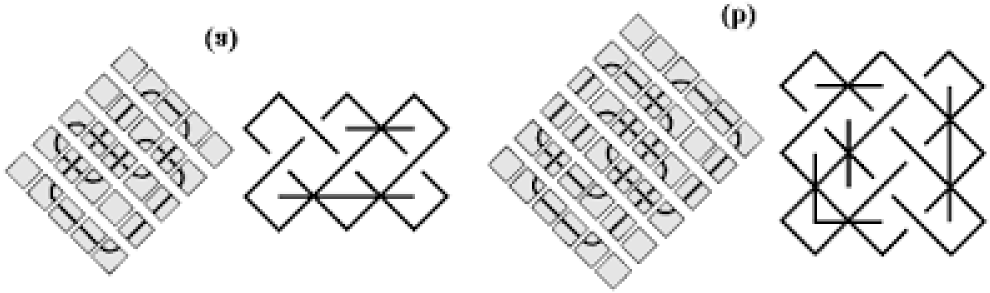


Figure 7: The figure-8 knot and Borromean rings arising from mirror curves.

6 Knots in art

Knots appear in the artwork and literature of individuals and cultures throughout the work. You may have heard the legend from the Hellenistic world about Alexander the Great: There was an enormous knot of rope in the ancient city Gordium (present-day Turkey), and a prophesy that whoever could undo this *Gordian knot* would be the ruler of all Asia. When Alexander came with his Macedonian-Greek army into the city, after pondering on the problem for a moment, he decisively drew his sword and hacked the knot to pieces.

Figure 8: *Alexander the Great cuts the Gordian Knot* by Jean-Simon Berthélemy (1743–1811).

Whether or not he cheated, Alexander would go on to conquer much of what the Greeks knew as Asia, but would die in murky circumstances, aged only 32, shortly thereafter.

6.1 Traditional artwork

Celtic culture: Representation of elaborate (and typically alternating) knots is closely associated with artwork of the Celtic world, especially in Ireland and Great Britain. The earliest examples of these designs first appear around the 4th century C.E., possibly under the influence of artistic knotwork from the Roman world.

(a) *Great Pavement*, Roman mosaic in Woodchester, England, 325 C.E.

(b) *Book of Kells*, c. 800 C.E.

(c) Carpet page from the Lindisfarne Gospels, c. 715-720 C.E.

(d) Stone Celtic cross.

Figure 9: Celtic artwork with knot designs.

Tamil culture: The Tamil people of South India use rice powder to create intricate patterns, called *kolam*, along the lines of the mirror curve construction from §5.3 at the thresholds of dwellings for special observances.

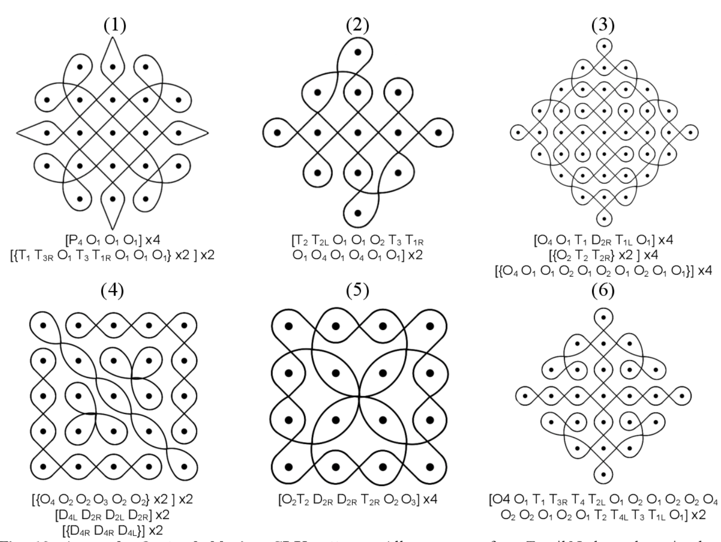


Figure 10: Tamil *kolam*.

Tchokwe culture: The Tchokwe people of southwestern Africa pass a particular artistic style of sand drawing down through the generations. The designs are, as with those of Tamil kolam, related to the mirror curves. Successive generations invest significant time and effort to learn a "vocabulary" of patterns and an algorithm for composing them into a kind of pictorial language.

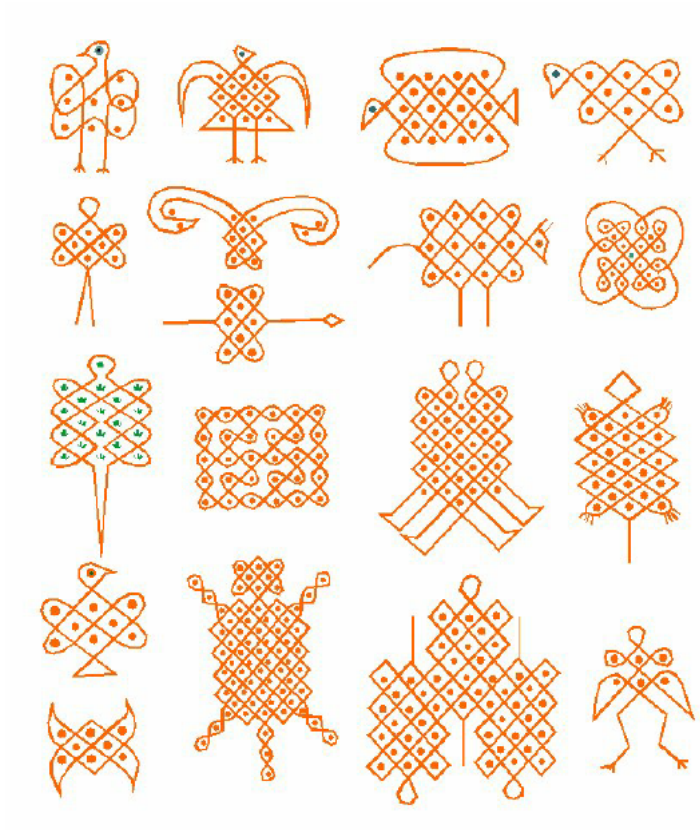


Figure 11: Tchokwe sand drawings.

7 Project Ideas

1. Use Reidemeister moves to produce a wild, unrecognizable representation of the unknot.
2. Build your own knot using any of the construction techniques from section 5.