What is incompleteness?

Art of Mathematics, Summer 2024

July 10, 2024

1 Introduction

Last class, we talked about the search for the "ground truth" of mathematics from the two historical perspectives of formalism (David Hilbert's) and logicism (Bertrand Russell's). When David Hilbert presented the formalist axioms for his new geometric system in the 1890s, he claimed that this system was **complete**—any true statement formulated in terms of its axioms could be proven within that system. Hilbert's claim marked a new philosophical way of thinking about math in terms of the language used to describe it. For example, if a biologist has a discussion with someone about biology, she is not actually *doing* biology. This is true of any science, with the notable exception of mathematics.

Hilbert's claim about the completeness of his geometric axioms illustrates two different "flavors" of mathematical statements: a statement **in** a mathematical system, like a geometric theorem, versus a statement **about** a mathematical system, like the completeness of his geometric axioms. The latter can be thought of as a "meta-mathematical" statement, and the key difference between math and other sciences is that meta-mathematical statements can be formulated *in the language of mathematics*. In other words, a mathematician who engages in a meta-mathematical discussion is actually *doing* mathematics.

In the early 1920s, Hilbert issued a challenge to mathematicians to find an axiomatic system for arithmetic, which became known as "Hilbert's Program." In particular, accomplishing the challenge meant proving two things:

1. The axiomatic system is **complete**, i.e. every true statement formulated mathematically can be proven within the system.

2. The axiomatic system is **consistent**, i.e. the set of axioms cannot lead to a paradox.

Thanks to the publication of Russell and Whitehead's *Principia Mathematica*, Hilbert could now take advantage of predicate logic to write mathematics in a formal language of symbols. For example, he could now symbolically express the mathematical statement

$$\textit{For any number } y\textit{, there is a number } x \textit{ that is larger than } y. \iff \forall y\ \exists x(Lxy),$$

where L represents the predicate relation "larger than," so that Lxy is interpreted as "x has the relation of 'larger than' to y." More crucially, he could also express metamathematical statements, such as

$$\textit{There is an } x \textit{ that is the proof of } y. \iff \exists xy(Dxy),$$

where D is the predicate relation "is the proof of," so that Dxy is interpreted as "x has the relation 'is the proof of' to y," or more simply "y is provable." Hilbert now hoped to achieve proofs of an axiomatic system's completeness and consistency in the language of the system itself.

2 Hilbert's Program

The mathematicians who were seeking to accomplish Hilbert's challenge attempted to do so using the **Peano axioms**, named for Giuseppe Peano who proposed them in his 1889 book *Arithmetices Principia, Nova Methodo Exposita* (translated as *The Principles of Arithmetic Presented by a New Method*). These axioms define arithmetic for the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

<u>**Peano Axioms**</u>:

1. 0 is a natural number.
2. Every natural number x is equal to itself.
3. For all natural numbers x and y, if $x = y$, then $y = x$.
4. For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$.
5. For all x and y, if x is a natural number and $x = y$, then y is a natural number.
6. For all natural numbers x, the successor of x is a natural number.
7. Distinct natural numbers have distinct successors.
8. 0 is not the successor of any natural number.
9. If a set of numbers contains 0 and the successor of every number in the set, then this set contains every natural number.

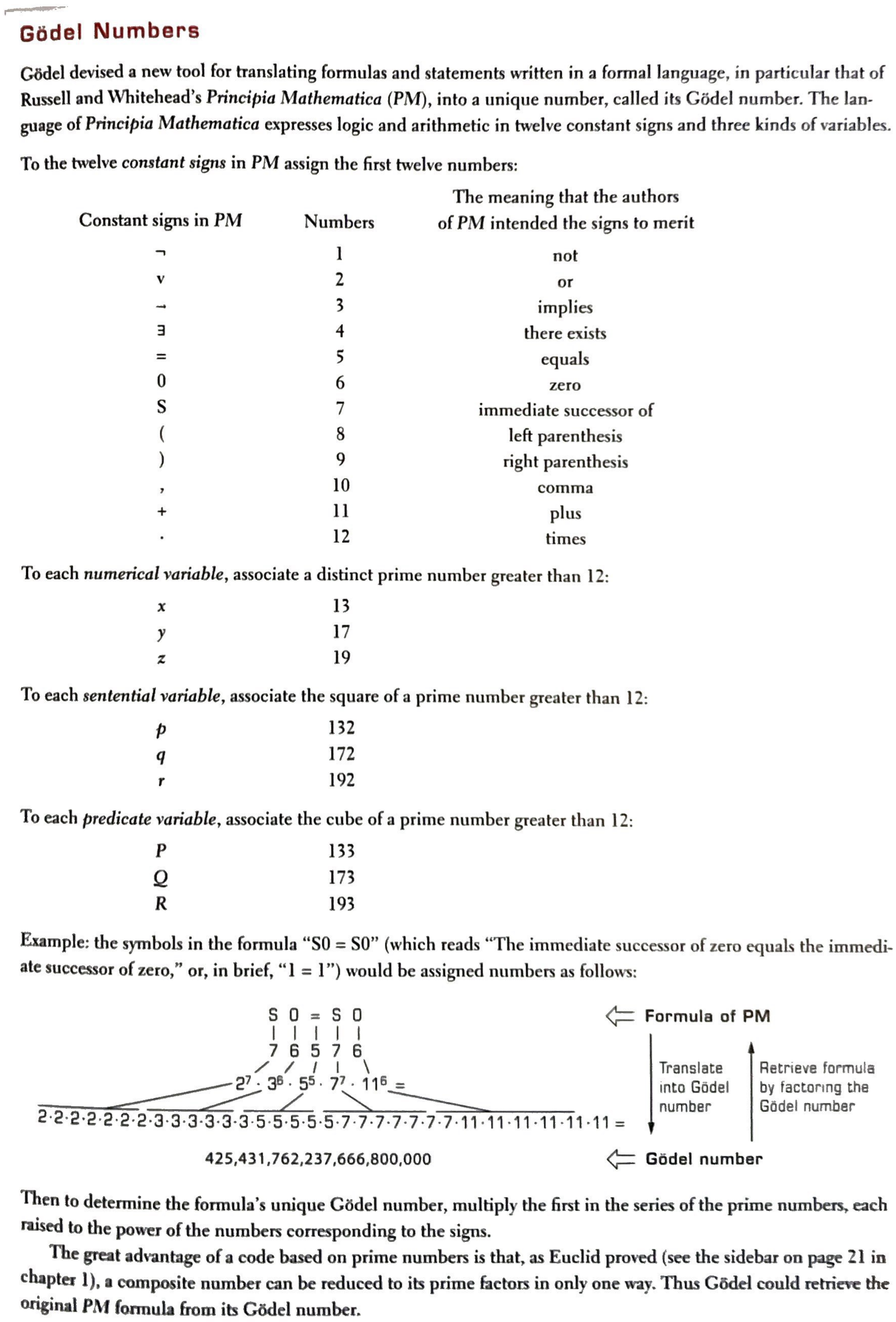
In *Principia Mathematica* in 1910, Bertrand Russell and Alfred Whitehead were able to show that the Peano axioms for arithmetic hold for a logical system constructed within predicate logic—proving that the laws of arithmetic could be formulated in the language of predicate logic. However, in order to do so, Russell and Whitehead had to rely on an "extra-logical" or metaphysical axiom, the **Axiom of Infinity**, which states that there exists at least one infinite set. But because of their results, mathematicians working on Hilbert's Program aimed to prove the completeness and consistency of the axioms for predicate logic in *Principia Mathematica*.

In 1929, Austrian-Hungarian mathematician Kurt Gödel proved that first-order predicate logic is complete, where the variables range only over individual numbers. When he attempted to prove the same for second-order predicate logic, which ranges over both individuals and sets of individuals like the natural numbers, he made a disturbing discovery: second-order predicate logic, and thus our axiomatic system of arithmetic, is **incomplete**.

3 Gödel's Incompleteness Theorem

Much to Hilbert's dismay, Gödel proved that no mathematical system capable of describing the natural numbers can avoid self-reference; the potential for paradox is inescapable. In other words, Hilbert's Program was impossible—no axiomatic system describing arithmetic can be both complete and consistent.

To accomplish his proof, Gödel developed a formulaic way to represent every statement in the vocabulary of *Principia Mathematica* as a unique number, known as a Gödel number (Figure 1). The premises of a given proof (taken as a whole) are assigned a unique Gödel number, and the conclusion of the proof is



Gödel Numbers

Gödel devised a new tool for translating formulas and statements written in a formal language, in particular that of Russell and Whitehead's *Principia Mathematica* (*PM*), into a unique number, called its Gödel number. The language of *Principia Mathematica* expresses logic and arithmetic in twelve constant signs and three kinds of variables.

To the twelve *constant signs* in *PM* assign the first twelve numbers:

Constant signs in *PM*	Numbers	The meaning that the authors of *PM* intended the signs to merit
¬	1	not
v	2	or
→	3	implies
∃	4	there exists
=	5	equals
0	6	zero
S	7	immediate successor of
(	8	left parenthesis
)	9	right parenthesis
,	10	comma
+	11	plus
·	12	times

To each *numerical variable*, associate a distinct prime number greater than 12:

x	13
y	17
z	19

To each *sentential variable*, associate the square of a prime number greater than 12:

p	132
q	172
r	192

To each *predicate variable*, associate the cube of a prime number greater than 12:

P	133
Q	173
R	193

Example: the symbols in the formula "S0 = S0" (which reads "The immediate successor of zero equals the immediate successor of zero," or, in brief, "1 = 1") would be assigned numbers as follows:

S 0 = S 0 ⇐ **Formula of PM**

7 6 5 7 6

$2^7 \cdot 3^6 \cdot 5^5 \cdot 7^7 \cdot 11^6 =$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 11 \cdot 11 \cdot 11 \cdot 11 \cdot 11 \cdot 11 =$

425,431,762,237,666,800,000 ⇐ **Gödel number**

Translate into Gödel number

Retrieve formula by factoring the Gödel number

Then to determine the formula's unique Gödel number, multiply the first in the series of the prime numbers, each raised to the power of the numbers corresponding to the signs.

The great advantage of a code based on prime numbers is that, as Euclid proved (see the sidebar on page 21 in chapter 1), a composite number can be reduced to its prime factors in only one way. Thus Gödel could retrieve the original *PM* formula from its Gödel number.

Figure 1: Courtesy of *Mathematics + Art: A Cultural History* by Lynn Gamwell

assigned another unique Gödel number. Gödel's method turns the question of whether a given conclusion follows from given premises into a numerical computation with Gödel numbers—what is *provable* in a formal axiomatic system becomes what is *computable* numerically.

In particular, Gödel shows there is a formula in the axiomatic system of *Principia Mathematica* which is necessarily true but cannot be deduced from the system's axioms. His argument evokes **self-reference**, reminiscent of the ancient Liar paradox considered in last week's lecture: the Cretan who declares, "All Cretans are liars."

First, Gödel demonstrates how to map *metamathematical* statements *about* formulas in *Principia Mathematica* to their Gödel numbers. Next, he shows how to construct a statement Q in the system of *Principia Mathematica* that is equivalent to a metamathematical statement (in Gödel number) with the interpretation that Q is not provable in the *Principia Mathematica* system. Note that Q is either true (provable) or false (not provable).

Suppose that Q is provable. Since Q is logically equivalent in the *Principia Mathematica* system to a metamathematical statement that Q is **not** provable, the system has an unacceptable logical contradiction. Therefore, we must assume that Q is not provable.

Then, suppose that Q is not provable. Because Q is equivalent in Gödel number to a metamathematical statement that Q is not provable, the statement "Q is not provable" is true. But now we have produced a true statement which cannot be proven. Therefore, the logical system of *Principia Mathematica* and hence our system of arithmetic, is **incomplete**. Gödel's proof shows that we are inherently limited in what we can formally prove within an axiomatic system.

4 Meta-Art: *L'art pour l'art*, or "Art for art's sake"

4.1 Cubism

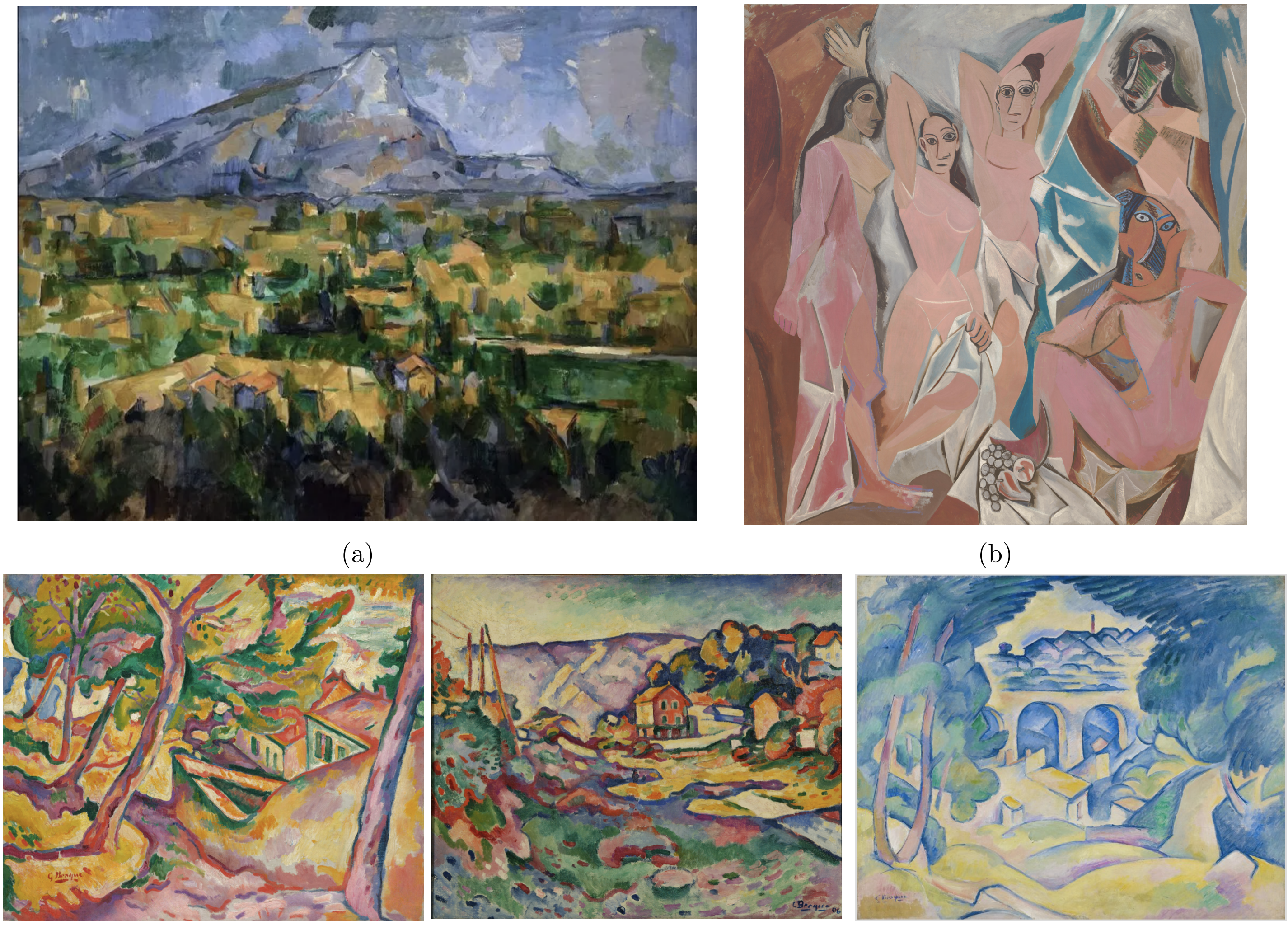
The Cubist style of art is attributed to Pablo Picasso (1881–1973) and Georges Braque (1882–1963), originating with Picasso's *Les Demoiselles d'Avignon* in 1907 (Figure 2(b)). The term Cubism was coined by the French art critic Louis Vauxcelles after seeing the landscapes that Braque had painted at L'Estaque (Figure 2(c-e)) in the style of Paul Cézanne (Figure 2(a)). Another major influence on Cubism was African art, after Picasso visited the ethnographic museum in the Palais du Trocadéro in Paris and began collecting African masks himself (Figure 3). Cubism was further developed by many other painters, including Fernand Léger, Robert and Sonia Delaunay, Juan Gris, Roger de la Fresnaye, Marcel Duchamp, Albert Gleizes, Jean Metzinger, and Diego Rivera.

Similar to David Hilbert's perspective on math, Cubist painters did not ascribe to the belief that art should mimic nature, or even that artists should use traditional techniques like perspective. Instead, they wanted to emphasize the material of which art is made–the 2-D canvas, shapes, colors–and to reduce objects into geometric components with multiple, sometimes competing, vantage points.

4.2 Surrealism

From "Surrealism" by James Voorhies in *Heilbrunn Timeline of Art History* by New York: The Metropolitan Museum of Art, 2000.

Surrealism originated in the late 1910s and early '20s as a literary movement that experimented with a new mode of expression called automatic writing, or automatism, which sought to release the unbridled imagination of the subconscious. Officially consecrated in Paris in 1924 with the publication of the *Manifesto of Surrealism* by the poet and critic André Breton (1896–1966), Surrealism became an international intellectual and political movement. Breton, a trained psychiatrist, along with French poets Louis



(a) (b)

(c) (d) (e)

Figure 2: (a) Paul Cézanne, *Mont Sainte-Victoire*, 1902-04. (b) Pablo Picasso, *Les Demoiselles d'Avignon*, 1907. (c) Georges Braque, *Landscape at L'Estaque*, 1906. (d) Georges Braque, *Landscape at L'Estaque*, 1906. (e) Georges Braque, *The Viaduct at L'Estaque*, 1907.

Aragon (1897–1982), Paul Éluard (1895–1952), and Philippe Soupault (1897–1990), were influenced by the psychological theories and dream studies of Sigmund Freud (1856–1939) and the political ideas of Karl Marx (1818–1883). Using Freudian methods of free association, their poetry and prose drew upon the private world of the mind, traditionally restricted by reason and societal limitations, to produce surprising, unexpected imagery.

Surrealist poets were at first reluctant to align themselves with visual artists because they believed that the laborious processes of painting, drawing, and sculpting were at odds with the spontaneity of uninhibited expression. However, Breton and his followers did not altogether ignore visual art. They held high regard for artists such as Giorgio de Chirico (1888–1978), Pablo Picasso (1881–1973), Francis Picabia (1879–1953), and Marcel Duchamp (1887–1968) because of the analytic, provocative, and erotic qualities of their work. In 1925, Breton substantiated his support for visual expression by reproducing the works of artists such as Picasso in the journal *La Révolution Surréaliste* and organizing exhibitions that prominently featured painting and drawing.

The visual artists who first worked with Surrealist techniques and imagery were the German Max

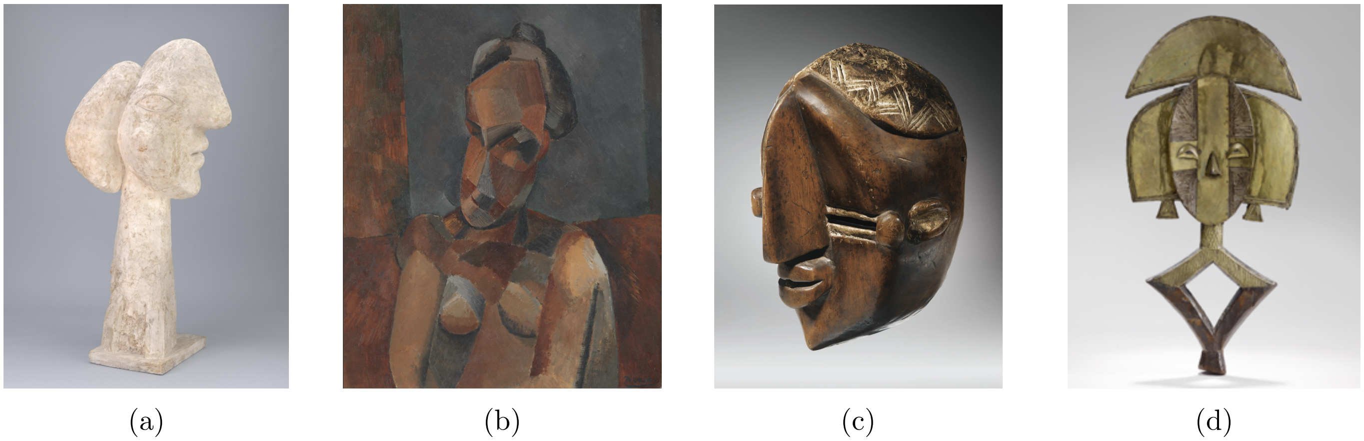


Figure 3: (a) Pablo Picasso, *Head of a Woman*, 1932. (b) Pablo Picasso, *Bust of a Woman*, 1909. (c) Mask, Democratic Republic of Congo, Lwalwa culture, Ethnographic museum in Paris. (d) Anthropomorphic Mask, Ivory Coast, Dan culture, Pablo Picasso's personal collection, before 1966.

Ernst (1891–1976), the Frenchman André Masson (1896–1987), the Spaniard Joan Miró (1893–1983), and the American Man Ray (1890–1976). Masson's free-association drawings of 1924 are curving, continuous lines out of which emerge strange and symbolic figures that are products of an uninhibited mind. Breton considered Masson's drawings akin to his automatism in poetry. Miró's *Potato* of 1928 uses comparable organic forms and twisted lines to create an imaginative world of fantastic figures.

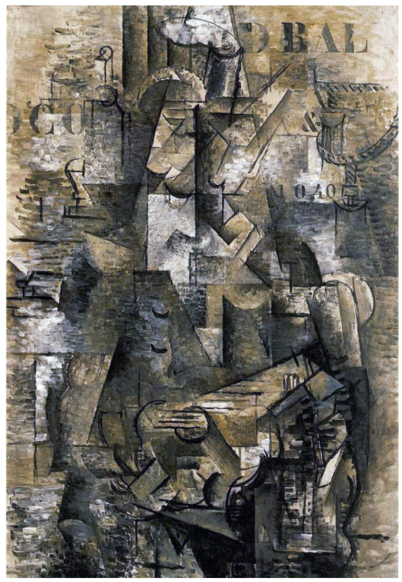
About 1937, Ernst, a former Dadaist, began to experiment with two unpredictable processes called decalcomania and grattage. Decalcomania is the technique of pressing a sheet of paper onto a painted surface and peeling it off again, while grattage is the process of scraping pigment across a canvas that is laid on top of a textured surface. Ernst used a combination of these techniques in *The Barbarians* of 1937, a composition of sparring anthropomorphic figures in a deserted postapocalyptic landscape that exemplifies the recurrent themes of violence and annihilation found in Surrealist art.

In 1927, the Belgian artist René Magritte (1898–1967) moved from Brussels to Paris and became a leading figure in the visual Surrealist movement. Influenced by de Chirico's paintings between 1910 and 1920, Magritte painted erotically explicit objects juxtaposed in dreamlike surroundings. His work defined a split between the visual automatism fostered by Masson and Miró (and originally with words by Breton) and a new form of illusionistic Surrealism practiced by the Spaniard Salvador Dalí (1904–1989), the Belgian Paul Delvaux (1897–1994), and the French-American Yves Tanguy (1900–1955).

In 1929, Dalí moved from Spain to Paris and made his first Surrealist paintings. He expanded on Magritte's dream imagery with his own erotically charged, hallucinatory visions. In *The Accommodations of Desire* of 1929, Dalí employed Freudian symbols, such as ants, to symbolize his overwhelming sexual desire. In 1930, Breton praised Dalí's representations of the unconscious in the *Second Manifesto of Surrealism*. They became the main collaborators on the review *Minotaure* (1933–39), a primarily Surrealist-oriented publication founded in Paris.

The organized Surrealist movement in Europe dissolved with the onset of World War II. Breton, Dalí, Ernst, Masson, and others, including the Chilean artist Matta (1911–2002), who first joined the Surrealists in 1937, left Europe for New York. The movement found renewal in the United States at Peggy Guggenheim's gallery, *Art of This Century*, and the Julien Levy Gallery. In 1940, Breton organized the fourth International Surrealist Exhibition in Mexico City, which included the Mexicans Frida Kahlo (1907–1954) and Diego Rivera (1886–1957) (although neither artist officially joined the movement).

(a) Picasso, *Ma Jolie*, 1911.



(b) Braque, *The Portuguese*, 1911.

(c) Juan Gris, *Still Life with Checked Tablecloth*, 1915.

(d) Diego Rivera, *The Cafe Terrace*, 1915.

(e) Sonia Delauney, *Flamenco Singer*, 1915.

(f) Fernand Léger, *Mechanical Elements*, 1920.

Figure 4: Examples of Cubism.

(a)

(b)

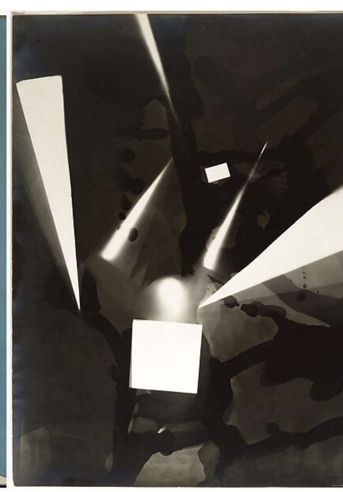
(c)

(d)

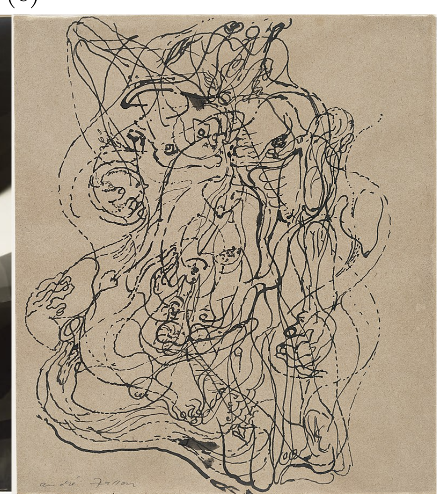
(e)

(f)

(g)



(h)

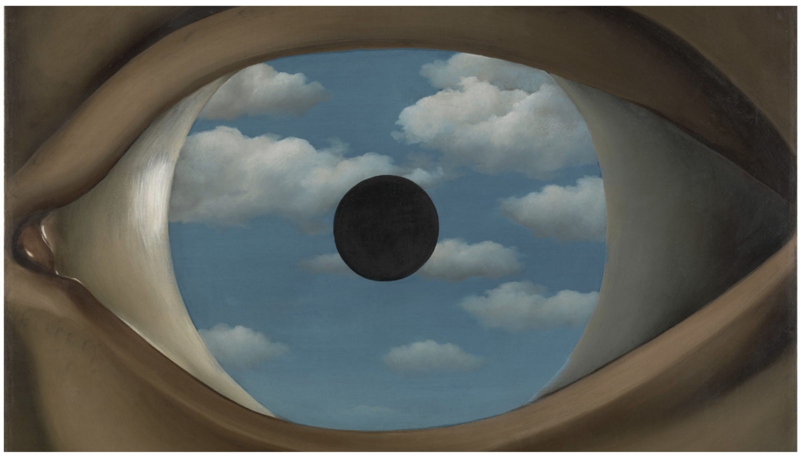


(i)

Figure 5: (a) Giorgio de Chirico, *The Jewish Angel* 1916. (b) Joan Miró, *Potato*, 1928. (c) Max Ernst, *Gala Eluard*, 1924. (d) Max Ernst, *The Barbarians*, 1937. (e) Salvador Dalí, *The Accomodations of Desire*, 1929. (f) Hans Bellmer, *The Doll*, 1934-35. (g) Pablo Picasso, *Nude Standing by the Sea*, 1929. (h) Man Ray, *Rayograph*, 1923-28. (i) Andre Masson, *Automatic Drawing*, 1924.

(a)

(b)



(c)

(d)

(e)

(f)

(g)

Figure 6: René Magritte paintings. (a) *The Lovers II*, 1928. (b) *Treachery of Images*, 1929. (c) *The False Mirror*, 1928. (d) *Golconda*, 1953. (e) *The Human Condition*, 1933. (f) *The Therapist*, 1937. (g) *The Healer*, 1967.