What is formalism?

Art of Mathematics, Summer 2024
Texas Prison Education Initiative
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1 Introduction

For over two thousand years, our notion of geometry was completely determined by what the ancient Greeks had postulated around 300 BC, called **Euclidean geometry** after the mathematician Euclid, one of the most prolific mathematicians in history. In his treatise *Elements*, Euclid put forth 5 axioms from which he believed all other geometric properties could be deduced through the process of logic. Four of these axioms were so "obvious," or self-evident, that any conceivable geometry should satisfy them:

1. A straight line may be drawn between any two points.
2. Any terminated straight line may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.

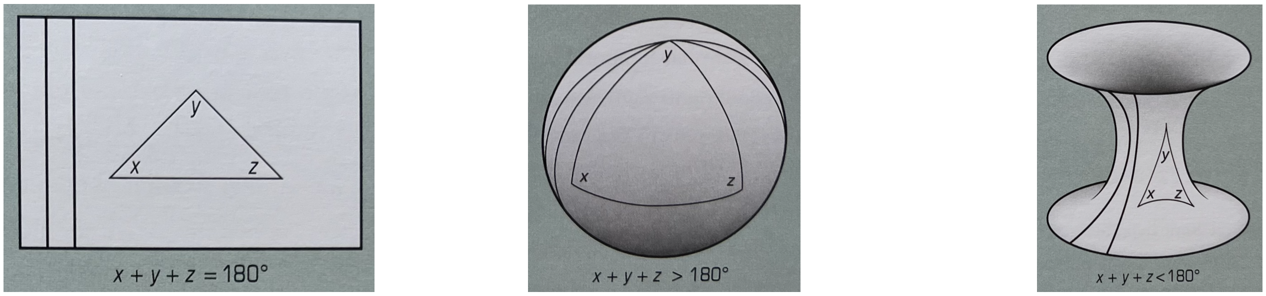
But the fifth axiom was a different sort of statement that drew skepticism from the moment it was proposed, precisely because it was not self-evident like the previous axioms:

5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.

Because it is much harder to describe than the others, this axiom seemed more like a theorem (something to be deduced or proved) than a self-evident declaration. But Euclid had failed to deduce it from the other four, so he included it as a fifth axiom because he needed it. As an important example, the fifth axiom was used to prove one of Euclid's most famous theorems: *The angles of a triangle add up to 180 degrees.*

Over time, mathematicians came up with simpler statements of the fifth axiom, for instance:

5'. For any point not on a given line, there is exactly one line through that point which does not intersect the line.



(a) Euclidean geometry (b) Spherical geometry (c) Hyperbolic geometry

Figure 1: Euclidean versus non-Euclidean geometries where Euclid's 5th axiom does not hold. Figures courtesy of "Mathematics + Art" by Lynn Gamwell.

This version of the fifth axiom is known as the *parallel postulate*. While it is simpler to understand than Euclid's original formulation, it was no easier to deduce from the other four axioms. This deduction remained an open question up to the 19th century, when mathematicians finally proved that Euclid's fifth axiom does not follow from the first four. Notably, this discovery led to the development of *non-Euclidean* geometries by assuming the negation of the fifth axiom (see Figure 1(b) and 1(c)). The discovery also led to a crisis in mathematics about what we know and what we can prove with absolute certainty. As a result, two important figures—mathematician David Hilbert and logician Bertrand Russell—sought to build a secure foundation for mathematical certainty. David Hilbert worked toward this goal by severing the ties between geometry and the natural world, working only from an axiomatic structure independent of our perception of reality—the basis of *formalism*. Bertrand Russell took a different approach, often referred to as *logicism*, believing that mathematics should be built up from the laws of logic informed by our natural language.

Along with his senior colleague Gottlob Frege, Bertrand Russell sought to create a system of rules to govern both mathematical language and our natural language. For example, Russell and Frege believed the arithmetic symbol "+" should represent the mathematical operation "*plus*" as well as the word "*and*" in everyday speech; their rules for the arithmetic operation were intermixed with natural language. David Hilbert instead formulated rules for manipulating the symbol "+" without any mention of "plus" or "and," attempting to keep his rules completely abstract. In Hilbert's formalist perspective, the statement "$5+7$" should only be interpreted as symbols and the rules governing how they're arranged—the *form* the expression takes. The quantities themselves should not matter: for Hilbert, "+" was just a formal relation between two numerals representing some abstract rule for manipulating those numerals. But Russell and Frege wanted to unify the meaning of "plus" in math with the meaning of "and" in natural speech, aiming to write rules that could capture both meanings. Despite their different philosophies, Hilbert's and Russell's objective was the same: develop a system of valid reasoning to derive scientific truths.

2 History of Formalism and Logicism

The first recorded attempts to work out a system of valid reasoning were by Aristotle in the 4th century, known as **deductive logic**, which he used in debates. He determined the truth of arguments by the truth of the premises from which those arguments could be deduced, as illustrated in the following example.

All people are mortal. Socrates is a person. Therefore, Socrates is mortal.

In general, Aristotle used letters like A, B, and C for categories, so that his logical arguments took the form: "All A are B. All B are C. Therefore, all A are C."

In the 17th century, mathematician Gottfried Leibniz recognized the importance of using precise language in science and began creating a "universal language" of mathematics to overcome the ambiguities in our natural language. His efforts had a lasting impact and influenced what is known as **propositional logic**. In propositional logic, letters P and Q represent *propositions*, or assertions that are either true or false. *Connectives* relate propositions to each other:

Connective	Meaning	Example	Meaning
$\wedge$	and	$P \wedge Q$	P and Q are true.
$\vee$	or	$P \vee Q$	P or Q is true.
$\rightarrow$	if ..., then ...	$P \rightarrow Q$	If P is true, then Q is true.
$\leftrightarrow$	if and only if	$P \leftrightarrow Q$	P is true if and only if Q is true.
$\neg$	not	$\neg P$	The negation of P ("not P") is true.

The truth of propositions is used to determine the truth of inferences. For example, consider the following true propositions: let P represent the proposition "All people are mortal," let Q represent the proposition "Socrates is a person," and let R represent the proposition "Some roses are red." Then the following assertion is a true inference in propositional logic:

$$\textit{If people are mortal and Socrates is a person, then some roses are red.} \iff P \wedge Q \rightarrow R.$$

A major drawback of propositional logic is that we cannot "get inside" the propositions to see how they relate to each other logically, and the resulting inferences may be non-intuitive.

This problem was partially solved by the introduction of **predicate logic** by Gottlob Frege in the 19th century. He developed a symbolic system to express the logical structure of a sentence, using symbols to represent predicates: for instance, in the sentence "Socrates is mortal," the predicate would be "is mortal." In addition to the connectives of propositional logic, he saw a need for *quantifiers* in logic which could further characterize propositions: $\forall$, meaning "for every," and $\exists$, meaning "for some" or "there exists." Frege used lowercase letters for variables (x, y, z) and constants (a, b, c). Capital letters represented predicates; for example, if P represents "is a person," G represents "is Greek," and M represents "is mortal," we can express the following sentences in predicate logic:

$$\textit{There exists a person who is Greek.} \iff \exists x(Px \wedge Gx)$$
$$\textit{There exists a person who is not Greek.} \iff \exists x(Px \wedge \neg Gx)$$
$$\textit{All Greeks are mortal.} \iff \forall x(Gx \rightarrow Mx)$$

Frege wanted to prove that all of mathematics—in particular arithmetic, what he considered the core of mathematics—could be derived from logical primitives like $\neg$ or $\forall$. He relied on what is known as *set theory*, introduced by the 19th century German mathematician Georg Cantor. For instance, in set theoretic terms, Frege considered the number "1" to be the set, or collection, of every set that has a single member: $1 = \{\{a\}, \{b\}, \{c\}, ...\{1\}, \{2\}, \{3\}, ...\}$. He showed that our basic laws of arithmetic, known as Peano's axioms, can be derived using only predicate logic from the principle of numerical equality: numbers, thought of as sets, are equal if and only if there is a one-to-one correspondence between the members of the sets. For instance, 1 is not equal to 2 from a set-theoretic perspective because there are more sets with two members than sets with one member, i.e. the set of all sets with two members is bigger than the set of all sets with one member, so 2 is bigger than 1.

However, there was a problem with Frege's argument treating numbers as sets. In 1902, he received a letter from his colleague Bertrand Russell who found a contradiction in Frege's work, *Basic Laws of Arithmetic*, right before Frege was about to publish the second volume of it. Russell asked whether Frege had considered "the set of all those sets not members of themselves." Does this set include itself? If it does, then it also doesn't. But if it doesn't, then it also does. This logical paradox could be deduced from Frege's published axioms, and Frege was forced to modify his previous work.

The problem with Frege's original argument is one called **self-reference** that has been known since at least the 1st century AD when the Apostle Paul wrote, "A prophet of their own said, 'All Cretans are liars.'" The prophet's statement is self-referential: if he is telling the truth, then he is also a liar. Russell's paradox shows that there are mathematical sets which require very careful treatment, such as the set of all imaginable mathematical objects; Russell would later publish his own work that did not allow for any self-reference in definitions. We will consider this point more deeply next week when we talk about **incompleteness**.

In 1910, Russell and his colleague Alfred Whitehead published *Principia Mathematica* in which they proved that arithmetic is reducible to logic, i.e. all numbers and numerical operations can be constructed from logical primitives. Critically, Whitehead and Russell had to invoke three key axioms to do this, which their critics disputed as provably true: the Axiom of Infinity, the Axiom of Choice, and the Axiom of Reducibility. However, as we'll see in next week's lecture, it would later be shown that their set of axioms was **incomplete**, meaning that there is some true formula which cannot be deduced from Whitehead and Russell's axioms.

3 Formalism in Art

Late 19th century artists experienced a similar crisis to that of 19th century mathematicians about the foundation of their field. They started asking what constituted art—rather than basing their paintings on real-life objects, artists began to experiment with color and form to create abstract paintings. The formalist approach to a mathematical foundation had a profound impact on the art world, beginning with Russian Constructivism in the 1910s, spreading westward in the 1920s and globally after 1945.

3.1 Russian Constructivism

In 1914, young Russian painter Vladimir Tatlin visited Pablo Picasso's studio in Paris and saw his collage works (*papiers collés*, Figure 2). Inspired, Tatlin returned to Russia and began working on his own 3-D collages, arranging simple shapes cut from sheet metal to emphasize the construction of his sculptures (Figure 3(a),(b)). His work was featured in the "Last Futurist Exhibition of Paintings 0.10 (zero-ten)," alongside fellow Russian painter Kazimir Malevich (Figure 3(c)). Other prominent Russian Constructivist artists include El Lissitzky (Figure 3(d)) and Alexander Rodchenko, who often used a ruler and compass to create his designs (Figure 3(e)). These avant-garde artworks emphasize engineering over painterly concerns like composition, as these artists wanted to bridge art with their Communist ideals of serving the masses; hence, math and science were prioritized above artistic expression.

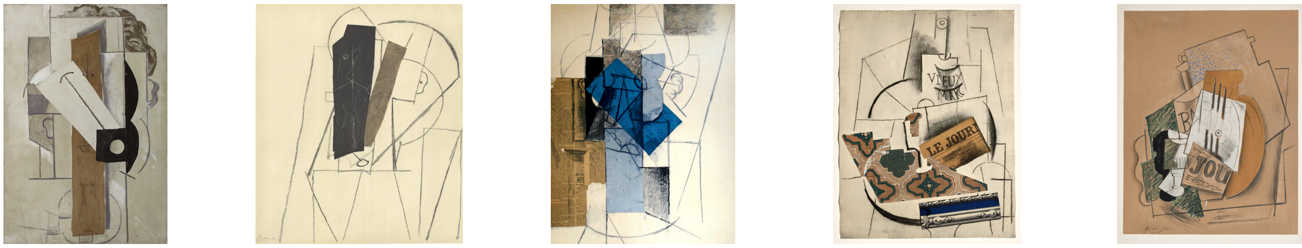
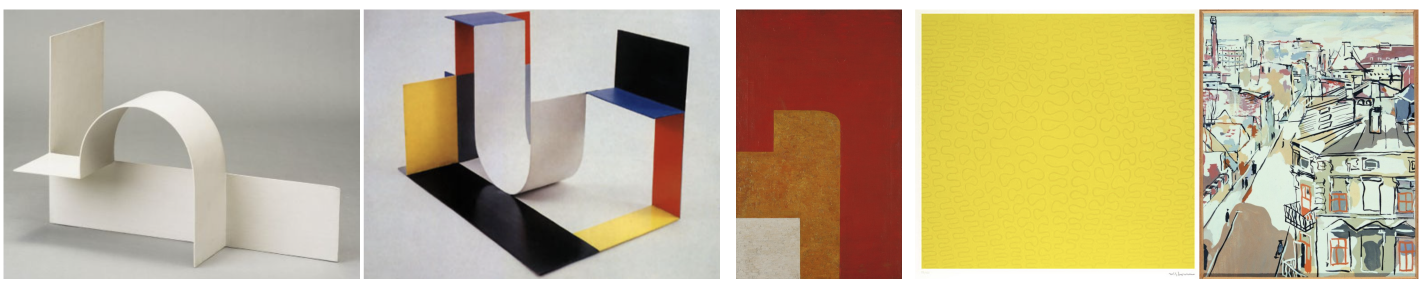


Figure 2: Picasso, *Papiers Collés*, 1910-1914.

(a) (b) (c) (d) (e)

Figure 3: Russian Constructivism. **(a)** Tatlin, *Corner Counter-Relief*, 1914-15. **(b)** Tatlin, *Tatlin's Tower*, 1919. **(c)** "The Last Futurist Exhibition of Painting 0.10," showing Malevich, 1915-16. **(d)** Lissitzky, *Study for "Proun" 8 Stellungen*, 1923. **(e)** Rodchenko, *Construction No.95*, 1919.



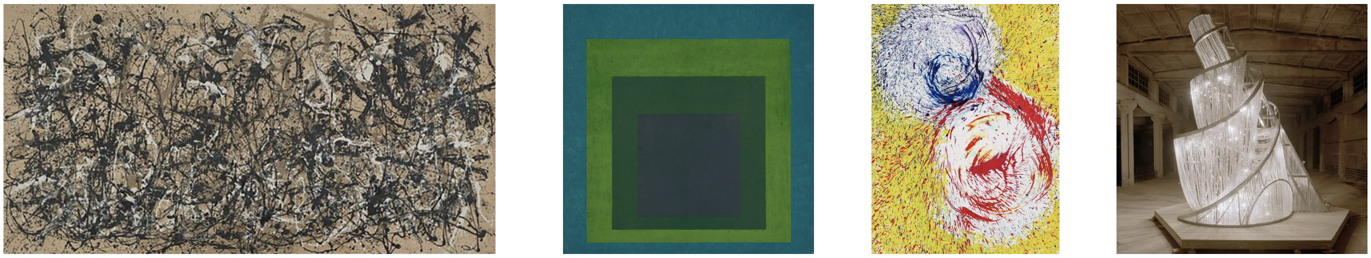
(a) (b) (c) (d) (e)

Figure 4: Early Formalism. **(a)** K. Kobro, *Space Composition 3*, 1928. **(b)** K. Kobro, *Space Composition 4*, 1929. **(c)** W. Strzeminski, *Kompozycja architektoniczna*, 1929. **(d)** W. Strzeminski, *Unist Composition*, 1931. **(e)** W. Strzeminski, *Lodz Landscape*, 1932.

3.2 Early Formalism

By the early 1920s, artists across Europe were producing abstract paintings, but many of them imbued their work with their emotions rather than pure objectivism. Polish artists like Katarzyna Kobro and Wladyslaw Strzeminski, influenced by Constructivism, saw the role of art as purely functional.

3.3 Modern Formalism



(a) (b) (c) (d)

Figure 5: Modern Formalism. **(a)** Jackson Pollock, *Mural*, 1950. **(b)** Josef Albers, *Soft Spoken*, 1969. **(c)** Dorothea Rockburne, *Singularity*, 1999. **(d)** Ai Weiwei, *Fountain of Light*, 2007.

4 Project Ideas

1. Draw a self-portrait using only geometric objects (lines, triangles, squares, circles, etc.)

(a) Picasso, *FEMME AU BÉRET ET À LA ROBE QUADRILLÉE (MARIE-THÉRÈSE WALTER)*, 1937.

(b) Picasso, *La Femme qui pleure (The Weeping Woman)*, 1937.

Figure 6

2. Create an abstract drawing paying homage to a single geometric shape (see Figure 3(c) or Figure 5(c) and (d)).

3. If you have the resources (i.e. paste, newspapers, magazines, etc.), create your own formalist collage (see Figure 2).