# What is randomness?

Art of Mathematics, Summer 2024

June 6, 2024

## 1 Introduction

Today we'll talk about randomness.

There have always been elements in works of art which could never have been precisely predicted by the artist. Think of Van Gogh painting (an impression of) what he saw in the sky in his *Starry Night*. What determined where the stars were located? Certainly scientists of today can say much about, say, the course will trace in the sky in a year, or why some are clustered together. In large part, however, as far as anyone knows, they ended up where they are simply by chance. There is a great deal in nature that happens according to chance and has been captured by artists, and that's to say nothing of all the randomness in the media of the art itself: little imperfections in the canvas or inconsistencies in the paint.

For most of history, artists largely accepted passively the randomness in nature, or even contrived to represent it more orderly, and sought to suppress unpredictable phenomena in the production of their work. Only very recently, in the last century, have artists deliberately endowed their work with elements of randomness. Techniques range from relinquishing precised control over paint application, as practiced famously by Jackson Pollock, to the use of computers to generate and arrange images at random.

## 2 Probability

> *"Probability is amongst the most important science, not least because no one understands it."*
> – Bertrand Russell (1872 – 1970), British philosopher, logician, mathematician, and historian.

You may have heard about the subjects of probability and statistics. Statistics is the analysis of how frequently past events have happened, while probability is the study of how likely it is for a future event to happen—in a sense, we use probability to analyze how well we can predict the future. In particular, probability is concerned with the idea of **randomness**.

So what is randomness, mathematically-speaking? We'll say that an event is **random** if the possible outcomes of the event occur with a certain *probability*, which we define next. A **probability** is a number between 0 and 1 that we assign to the possible outcome of a random event, which we can think of as an indicator of how likely a future event is to happen. A probability of 0 means there is no chance of the event happening, while a probability of 1 means the event will definitely happen.

In general, the probability of a specific outcome of a random event happening is defined as follows:

$$\text{Probability of outcome} = \frac{\text{\# of times the outcome can happen}}{\text{\# of total possible outcomes}}$$

**Example 1 (Coin Toss):** To make these things concrete, let's consider a familiar example: a (fair) coin toss. We'll say that the random event is a single coin toss, and we want to know the probability of the outcome that the coin comes up heads. First, what is the total number of possible outcomes? Since the coin can either come up heads or tails, there are 2 possible outcomes; let's call them $H$ and $T$ for short. Note: we usually express the possible outcomes in mathematical **set notation**, using curly braces: $\{H, T\}$. Now out of these total outcomes, the outcome of getting $H$ happens once, so

$$\text{Probability of } H = \frac{1}{2}.$$

Usually, to talk about random events more formally in mathematics, we introduce the concept of **random variables**, usually denoted by the variable $X$. Despite its name, a random variable is actually a function, which can take on any value in the set of possible outcomes of a random event. For example, let's say the random variable $X$ represents the outcome of a single coin toss, the random event from the previous example. Then either the coin comes up heads, so $X = H$, or it comes up tails, so $X = T$. Formally in the language of math, to say that the probability of the coin coming up heads is 1/2, we write

$$P(X = H) = \frac{1}{2}.$$

**Example 2 (Playing Cards):** Another common example used in teaching probability is the random event of drawing a card from a deck of cards. Consider a usual deck of 52 cards where there are 4 suits (hearts, spades, diamonds, clubs) having the values 2,...,10, Jack (J), Queen (Q), King (K), Ace (A). Suppose we are interested in the outcome of drawing a diamond. For this random event outcome to occur, we would need to draw any of the following cards in the suit of diamonds: 2,3,4,5,6,7,8,9,10,J,Q,K,A. So, there are 13 ways that we can draw a diamond card, out of 52 possible cards:

$$\text{Probability of drawing diamond} = \frac{13}{52} = \frac{1}{4}, \quad \text{or}$$

$$P(X = D) = \frac{1}{4},$$

where the random variable $X$ (representing the random event of drawing a card) takes the value $D$ for the specific outcome of diamonds.

Of course, we can ask more complicated questions about probability, but we would need to introduce more mathematical machinery. For instance, we could ask, "What is the probability of drawing a card that is an Ace (of any suit) and then drawing another card that is also an Ace?" To answer this question, we need to introduce the notion of **conditional probability**—that is, the probability than some event $X$ occurs given that event $Y$ has already happened.

**Example 3 (Playing Cards with Conditional Probabilities)**: Let's compute the probability that, if we are dealing 2 cards and the first card is an ace, that the second card is also an ace. Let $Y$ be random variable representing the event of drawing the first card, and let $X$ be the random variable representing the event of drawing the second card. Note that if these events occurred independently,

$$P(X = \text{Ace}) = \frac{4}{52} = \frac{1}{13}, \quad \text{and}$$

$$P(Y = \text{Ace}) = \frac{4}{52} = \frac{1}{13}.$$

But we are actually interested in a different quantity, called the conditional probability:

$$P(X = Ace|Y = Ace), \text{ often abbreviated to } P(X|Y),$$

which the probability that the second card is an Ace, given that the first card is an Ace. Intuitively, once you already "use up" an Ace, it should be less likely for you to draw another one. So, what is the probability of drawing a second Ace?

To answer, we'll need to formally define conditional probability:

**Definition (Conditional Probability):** Let two random events be denoted by $X$ and $Y$. If $P(Y) > 0$, then the **conditional probability of $X$ given $Y$** is

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)},$$

where $P(X \cap Y)$ is the probability that both events happen together.

For our previous example of drawing two cards, the probability $P(X \cap Y)$ represents the probability that both cards are Aces. Let's figure out how many total outcomes there are now: note that there are 52 options for the first card, but once we pick one, we then have 51 options for the second card. So, the total number of outcomes of drawing two cards is $52 \times 51$. To figure out the possible outcomes that give two aces: there are 4 possible Aces for the first card, and 3 possible Aces for the second card, for a total of $4 \times 3 = 12$ possible outcomes that give $X = Ace$ **and** $Y = Ace$. Then the probability of drawing two aces is

$$P(X \cap Y) = \frac{12}{52 \times 51}.$$

We already determined that the probability of drawing an Ace (either $X$ or $Y$) is $13/52 = 1/13$, so altogether

$$\begin{aligned} P(X|Y) = \frac{P(X \cap Y)}{P(Y)} &= \frac{12}{52 \times 51} \div \frac{4}{52} \\ &= \frac{12}{52 \times 51} \times \frac{52}{4} \\ &= \frac{3}{51}. \end{aligned}$$

Note that this probability is smaller than 4/52, which says that if the first card is an ace, it makes it less likely that the second card is also an ace, agreeing with our intuitive explanation above.

# 3 An Application of Probability: Random Walks

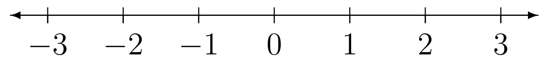
In math, a **random walk** describes a path that is formed by a sequence of random steps across a given space. Random walks occur frequently in nature; examples include:

- The path of a molecule as it travels in a liquid or a gas
- The path of an animal searching for food

- The price of stocks fluctuating in the stock market
- The financial status of a gambler at a casino

The term was first introduced in 1905 by Karl Pearson, a pioneer in the fields of probability and statistics.

To illustrate the idea, consider a number line of integers, and imagine a person standing on the 0 marker:




Now suppose at each step, the person flips a coin. If it comes up heads, the person takes a step to the right (+1); if it comes up tails, the person takes a step to the left (-1).

If we let this person continue walking infinitely, it can be shown that the person will step on every single marker. This result is known as **the gambler's ruin**. A gambler who has a finite amount of money will eventually lose when playing a fair game (with a 50-50 chance of winning) against a casino that has an infinite amount of money. The gambler's money will perform a random walk starting from their initial balance, but because that walk will eventually include every integer, it will reach zero at some point, and the game will be over.

The number line example is in 1-D: the person can only walk in two directions (left or right) along the line. But today we'll be interested in a 2-D walk: imagine a person walking randomly around a large (effectively infinite) city that is arranged in a square grid of sidewalks, like New York City. At every intersection, the person randomly chooses 1 of now 4 possible routes (including the one originally travelled from). Formally, this is a random walk on the set of all points in the plane with integer coordinates. We'll look at an artist in the next section (Kenneth Martin) who uses this idea to make geometric drawings, and you'll draw some yourself for your project this week!

# 4 Artwork

Figure 1: *No. 5, 1948*, Jackson Pollock (1912-1956), American abstract expressionist painter, best known for his "drip technique", also called "action painting" or "all-over painting"

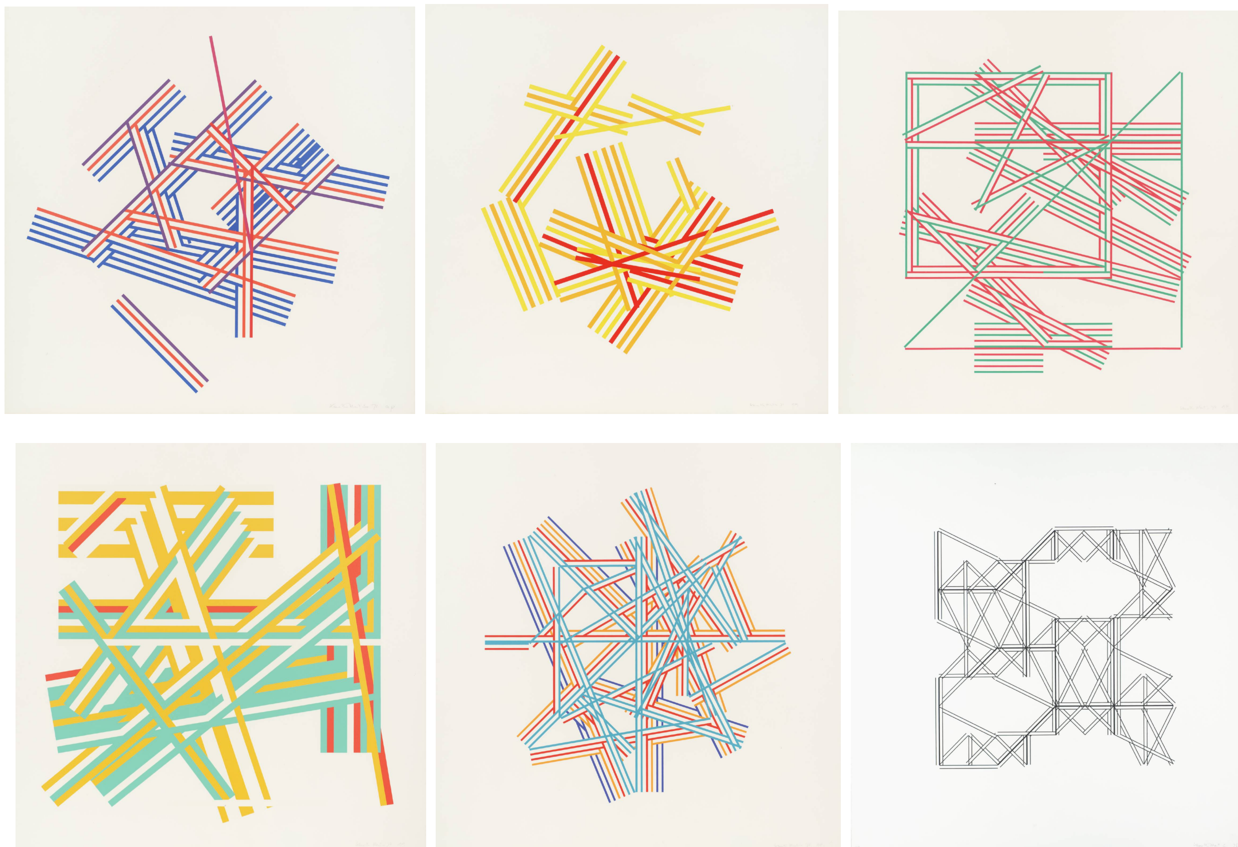


Figure 2: Series: *Chance and Order I-VI* (1969), Kenneth Martin (1905-1984), English painter and sculptor of the mid-20th century Constructivist revival.

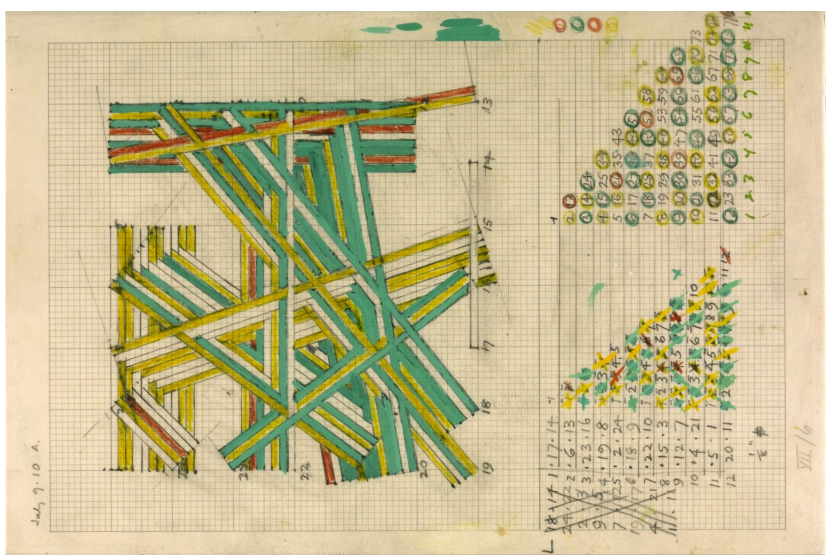
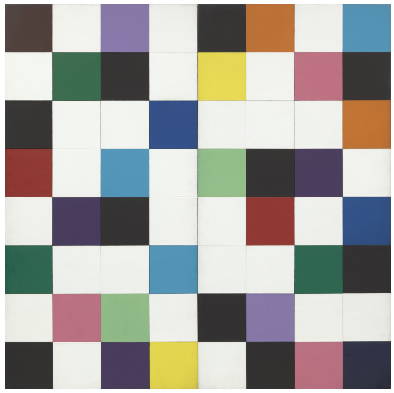


Figure 3: *Chance and Order Group VII Drawing* (1969), Kenneth Martin

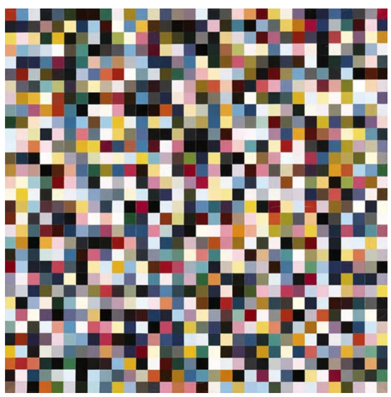
> *"The points of intersection on a grid of squares are numbered and the numbers are written on small cards and picked at random... A line is made between each successive pair of numbers as they are picked out."*

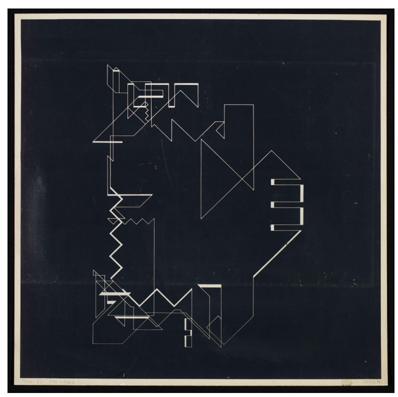
—Kenneth Martin



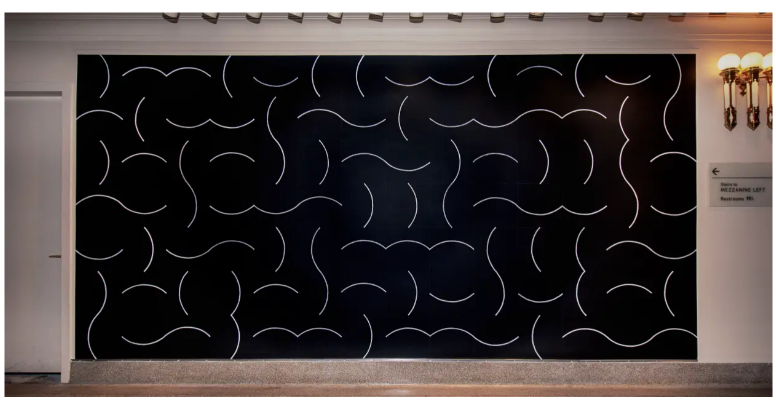
(a) (b)

Figure 4: Left to right: *Colors For a Large Wall*, *Colors Arranged By Chance* (1951), Ellsworth Kelly (1923-2015), American painter, sculptor and print-maker, associated with the minimalist movement.





(a) (b)



(c)

Figure 5: (a) *1024 Colors* (1973), Gerhard Richter (1932-), German visual artist with work ranging from the photo-realistic to highly abstract. (b) *P-18 (random walk)* (1970), Manfred Mohr (1938-), German visual artist, pioneer in computer art. (c) *Wall Drawing 357* (1981), Sol Lewitt (1928-2007), American painter and sculptor, prominent figure in the conceptual art and minimalist movements.

# 5 Project Ideas

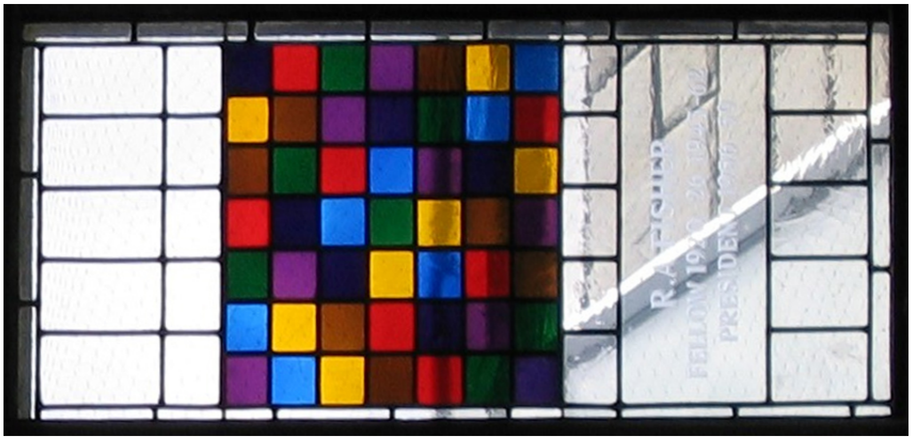


Figure 6: Stained glass window, in the form of a Latin square, in memory of British statistician R. A. Fisher, Gonville and Caius College, Cambridge.

1. First, form a $N \times N$ Latin square (like a chessboard, but colored in a special way described next). You'll color in the small squares of the grid from left to right, top to bottom, as follows: Put your colored pencils in a bag, or in a pile, and jumble them up. Draw a pencil at random (e.g. by closing your eyes if they're in a pile in front of you), and use it to color in the next square in your sequence. Now you may do any of the following:

   - Replace the pencil you've drawn, jumble up your pencils again, draw a new one and color your next square.
   - First jumble the pencils and draw a new one, and then replace the one you've previously drawn.
   - Jumble the pencils and draw a new one. Don't replace any pencils until you've run out.

   Proceed in any of the ways listed above until you've completed coloring your Latin square. You may also devise a "draw-and-replace" scheme of your own; just be sure to record a description to accompany your piece.

2. Random walks: follow Kenneth Martin's method of using random walks in a 2-D plane to create unique drawings.

3. Random art! Start drawing without any particular image in mind. Let your pencil(s) travel freely in random sequences and shapes.