What is infinity?

Art of Mathematics, Summer 2023

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1 Introduction: What is infinity?

Last time, we introduced the idea of different sizes of infinity.

First, we have the most "natural" numbers, which are in fact called the natural numbers, or sometimes counting numbers, represented by a boldfaced N ($\mathbb{N}$):

$$\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$$

This is an infinite set of numbers. For an intuitive proof, give me the largest natural number you can think of. Now add 1 to it. You have a new natural number, which is bigger than the biggest one you could think of. Repeat forever.

Now, for finite sets (a set in math just means a collection of things, usually written inside curly braces { ... }), it's easy to compare which one has more stuff in it. We call the size of a set its **cardinality**.

For example, the finite set $\{1, 2, 3\}$ has cardinality 3 because there are three things in it (1, 2, and 3). The finite set $\{1, 2, 3, 4, 5, 6\}$ has cardinality 6, etc. It's easy to see that the cardinality of the set $\{1, 2, 3, 4, 5, 6\}$ is bigger than the set $\{1, 2, 3\}$; to prove it, we'd match up the things in the set to show that the smaller set runs out of things first.

$$\begin{gathered}\{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3\} \\ 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \\ 4 \mapsto \textit{...uh oh}\end{gathered}$$

Note: we use $\to$ when we are talking about mapping sets, and $\mapsto$ when we are talking about mapping individual elements to each other.

We run out of the things in the smaller set that can be "mapped to" uniquely, so the set $\{1, 2, 3\}$ is smaller. By the way, it doesn't matter which thing maps to which thing, only that it's unique. So we could have also done

$$\begin{aligned} \{1,2,3,4,5,6\} &\to \{1,2,3\} \\ 4 &\mapsto 1 \\ 1 &\mapsto 2 \\ 6 &\mapsto 3 \\ 5 &\mapsto \textit{...uh oh} \end{aligned}$$

and we'd still find that $\{1, 2, 3\}$ is smaller. In math notation, we write $|\{1, 2, 3\}| = 3$ to denote its cardinality.

Now, to show that sets have the same cardinality, we need to find such a mapping between them, where every number in one set gets mapped to a unique number in the other set, and nobody gets left out.

Let's think about another infinite set and see if we can find such a mapping between it and $\mathbb{N}$; we'll first consider the integers, represented by $\mathbb{Z}$ (the Z is a reference to the German word "zahlen" which means "numbers"):

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

Notice that the integers actually contain the natural numbers inside of them; in math, we say that "the natural numbers are a proper subset of the integers" and we write it like this:

$$\mathbb{N} \subset \mathbb{Z}.$$

Something to note that will be important later; every set by definition is a subset of itself, but not a proper one! If S is any set, we'd write $S \subseteq S$, which is now true because of the tiny horizontal line that is a nod to an "=" symbol (since $S = S$ trivially).

Now, it *seems like* the integers should be bigger than the natural numbers (it seems like they should be twice as big, just like $\{1, 2, 3, 4, 5, 6\}$ is twice as big as $\{1, 2, 3\}$). **However, this kind of logic breaks down when we are dealing with infinite sets.** To show you, I propose the following mapping between them that follows all the rules: every member in one set goes to a unique member of the other set, and nobody gets left out.

Here's a possible map you could come up with to make this work (by the way, these kinds of maps are called **bijections**):

$$\begin{aligned} \mathbb{Z} &\to \mathbb{N} \\ 0 &\mapsto 0 \\ -1 &\mapsto 1 \\ 1 &\mapsto 2 \\ -2 &\mapsto 3 \\ 2 &\mapsto 4 \\ -3 &\mapsto 5 \\ 3 &\mapsto 6 \\ \textit{etc.} \end{aligned}$$

What is happening here? We map zero to zero. We map negative integers uniquely to odd natural numbers, and we map positive integers uniquely to even natural numbers. Nobody gets left out, on either side of the map. We have created a bijection between these two infinite sets; therefore, **they have the same cardinality**: $|\mathbb{N}| = |\mathbb{Z}|$ even though $\mathbb{N} \subset \mathbb{Z}$.

Note: $|\mathbb{N}|$ gets the special notation $\aleph_0$, pronounced "aleph naught." So, $|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$. Any infinite set with cardinality equal to $\aleph_0$ is said to be **countable**; otherwise, it is uncountable. In other words, infinite sets are very weird, and they disturbed mathematicians for centuries.

We now might think of the **rational numbers**, denoted $\mathbb{Q}$, which are all of the ratios a/b, where a and b are integers (with $b \neq 0$). Notice that if $b = 1$, we get all of the integers back, since $a/1 = a$ is an integer by definition. So we have the following proper subsets:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Perhaps surprisingly, $|\mathbb{Q}| = \aleph_0$ too. Why? Consider the following arrangement of the rational numbers:

$$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{4}{1}, \frac{4}{3}, \frac{-4}{1}, \frac{-4}{3} \ldots$$

Notice that every rational number will appear somewhere in the list above, and we can now map the rationals directly onto the natural numbers, one at a time, and nobody will be left out. So, even though $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$, we have shown that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$.

2 A Bigger Infinity

At this point, we should be asking ourselves: is there an infinite set that has a different cardinality than $\aleph_0$? In 1891, the famous German mathematician who invented set theory, Georg Cantor, proved that indeed there is. Here's how he constructed it.

Let S represent all possible infinite sequences of 0s and 1s (infinite binary sequences). Elements of S would then look like this:

$$\begin{aligned}
s_1 &= 0,0,0,0,0,0,0,\ldots \\
s_2 &= 1,1,1,1,1,1,1,\ldots \\
s_3 &= 0,1,0,1,0,1,0,\ldots \\
s_4 &= 1,0,1,0,1,0,1,\ldots \\
s_5 &= 1,1,0,1,0,1,1,\ldots \\
s_6 &= 0,0,1,1,0,1,1,\ldots \\
s_7 &= 1,0,0,0,1,0,0,\ldots \\
&\vdots \\
&\mathit{etc.}
\end{aligned}$$

The size of S is clearly infinite, as we can construct an infinite number of these (infinite, binary) sequences s_k, but is S countable; i.e. is $|S| = \aleph_0$? No!

Cantor showed with his famous **diagonalization proof** that we can in fact find an infinite, binary sequence s that does not appear in our set above of $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, \ldots\}$. How do we find the missing sequence s?

First, choose the first element of s to be the "opposite" of the first element of s_1. Since the first element of s_1 is 0, the first element of s would then be 1.

Next, choose the second element of s to be the "opposite" of the second element of s_2. Choose the third element of s to be the opposite of the third element of s_3, and repeat forever. In other words, the kth element of s should be "opposite" of the kth element of s_k in our list above. To see why this is called the diagonalization argument, we consider the following visualization of s:

$$
\begin{aligned}
s_1 &= (\underline{\mathbf{0}}, 0, 0, 0, 0, 0, 0, \ldots) \\
s_2 &= (1, \underline{\mathbf{1}}, 1, 1, 1, 1, 1, \ldots) \\
s_3 &= (0, 1, \underline{\mathbf{0}}, 1, 0, 1, 0, \ldots) \\
s_4 &= (1, 0, 1, \underline{\mathbf{0}}, 1, 0, 1, \ldots) \\
s_5 &= (1, 1, 0, 1, \underline{\mathbf{0}}, 1, 1, \ldots) \\
s_6 &= (0, 0, 1, 1, 0, \underline{\mathbf{1}}, 1, \ldots) \\
s_7 &= (1, 0, 0, 0, 1, 0, \underline{\mathbf{0}}, \ldots) \\
&\ldots \\
s &= (\underline{\mathbf{1}}, \underline{\mathbf{0}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{0}}, \underline{\mathbf{1}}, \ldots)
\end{aligned}
$$

By construction, s is a member of S that differs from each s_n, since their nth digits are different (in bold in the visualization above). So, s cannot appear in our list anywhere.

Next, Cantor then uses a **proof by contradiction** to show that S is uncountable. The proof starts by assuming that S is countable. Then all its elements can be counted, or numbered, as $s_1, s_2, s_3, \ldots$. We have just shown that we can find a sequence s, which is a member of S, that does not appear in our countable list $s_1, s_2, s_3, \ldots$. However, if S were countable, then every member of S, including this s, would be enumerated in the list. This contradiction implies that our original assumption is false, so S is uncountable. In other words, S has a larger cardinality than $\aleph_0$, what we previously thought of as "infinity."

This has HUGE ramifications for our number system, because we can actually show that the **real numbers are in bijection with this set** S. Recall that the real numbers, written $\mathbb{R}$, contain all of the numbers that we know and love which are not complex (or "imaginary"). So $\sqrt{2}$ is in $\mathbb{R}$, π is in $\mathbb{R}$–anything on the entire (real) number line we've seen before. Then we are saying here that $|S| = |\mathbb{R}| > \aleph_0$. We omit the proof, but it is based on decimal representations of real numbers written in binary (sequences of 0s and 1s like the members of S above).

3 An "In-Between" Infinity? The Continuum Hypothesis

We've now seen that the set of real numbers is "bigger" than the set of natural numbers, so it would make sense to ask if there is a set whose cardinality is "in between" that of the naturals and that of the reals.

This is an open question summarized in the continuum hypothesis, formulated by Cantor: **there is no set whose cardinality is strictly between that of the natural numbers and the real numbers.** He believed it to be true and tried for many years to prove it in vain.

While it is still unresolved, logicians in the 20th century made some disturbing findings in pursuit of the solution. Under the usual axioms, or accepted truths, that we rely upon in mathematics, the continuum hypothesis can neither be proven nor disproven.

4 Tiny Infinities: The Idea of Infinitesimals

We've talked before about how Archimedes got impressively close to the value of π by approximating the circumference of a circle. Let's look at it more closely.

Archimedes drew polygons with n sides that circumscribe (fit around) a circle, as well as polygons that inscribe (fit inside) a circle.

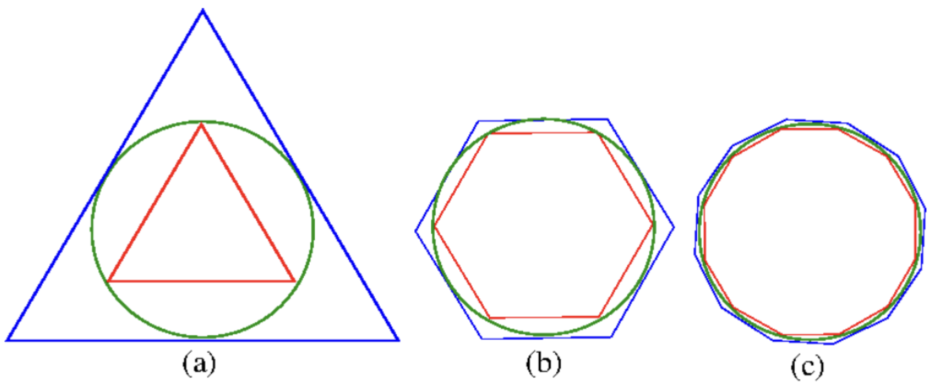


Figure 1: Polygons inscribed in and circumscribed around a circle with number of sides (a) $n = 3$, (b) $n = 6$, and (c), $n = 12$.

Each n-sided polygon inscribed in the circle has a smaller perimeter than the circumference of the circle; denote its perimeter by c_n.

Each n-sided polygon circumscribed around the circle has a bigger perimeter than the circumference of the circle; denote its perimeter by C_n.

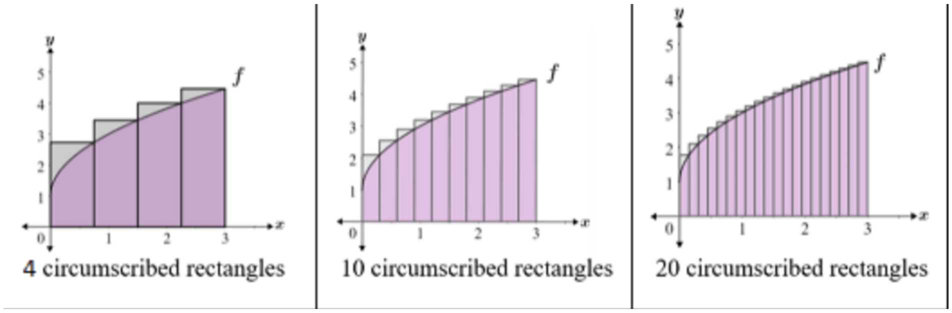
Let C denote the circumference of the circle. Then for each value of n, we have

$$c_n < C < C_n.$$

Notice how as n gets larger, the perimeters of the polygons get closer and closer to the true circumference of the circle. Archimedes used triangles to determine the perimeters of the polygons for large enough values of n to get a good approximation of C, which he used to determine π as the ratio of C/d, where d is the diameter of the circle. But what would happen if we went on forever with n-sided polygons? Would we ever reach the true value C? This is the idea behind **limits**.

Another place that limits appear in math is in finding the area of a complicated region. Let's say that you are an artist, and you need to know how much material you need for an art piece, which fits under the graph of the function $y = \sqrt{x} + 1$, or more formally, $f(x) = \sqrt{x} + 1$. How could you find the area of the piece?

We can steal Archimedes' idea, but we'll be less fancy and use successively skinnier rectangles circumscribed around our funny shape, instead of n-sided polygons. Why? Rectangles have a super easy formula to measure their area: length times width. This is the fundamental principle behind **integral calculus**. To find the area under a curve, we use n circumscribed rectangles, add up their areas, and then take the limit as n goes to infinity.



5 Projects

Create drawings inspired by either "big" infinities or "tiny" infinities, or both! To get you started, here are some examples again from M.C. Escher, who was inspired to sketch "the point of infinite smallness":

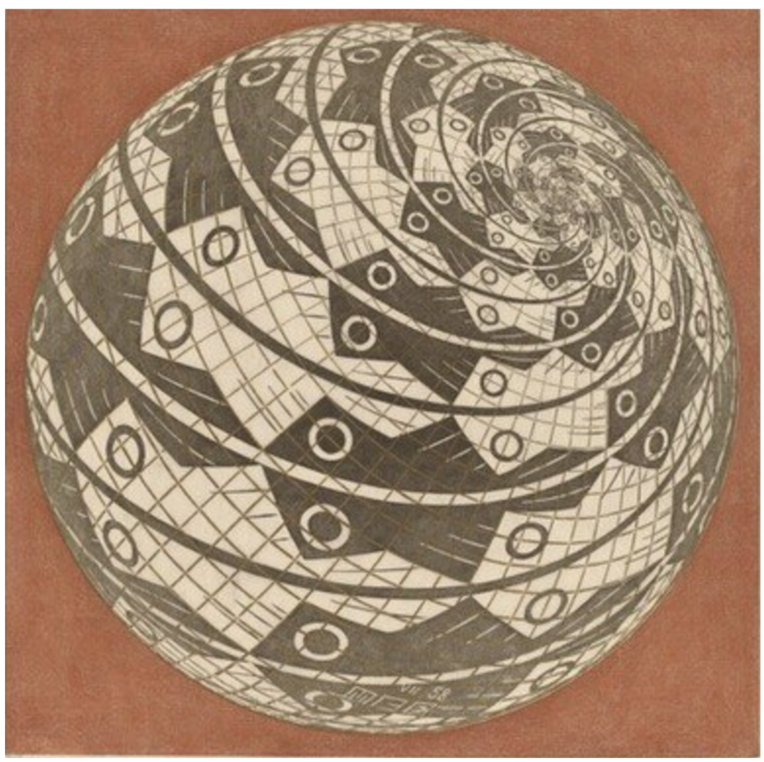


Figure 2: M.C. Escher (1958). A sphere with two poles and a network of longitudinal and latitudinal circles. Fish swim in a spiral from pole to pole, in alternate rows of black and white. Credit: SneakyArtist, "Escher and Infinity"

Figure 3: M.C. Escher, "The Path of Life II" (1958). Credit: SneakyArtist, "Escher and Infinity"

Escher was dissatisfied with his attempts at what he considered to be illogical limits, and he ended up corresponding with the famous mathematician Coxeter, who was a fan of Escher's work and introduced him to his own work on the Poincaré Disk:

In the 1905 treatise "Science and Hypothesis," Poincaré describes a world, now known as the Poincaré disk, in which space was Euclidean (satisfying the 5 axioms of Euclid), but which appeared to its inhabitants to satisfy the axioms of hyperbolic geometry:

"Suppose, for example, a world enclosed in a large sphere and subject to the following laws: The temperature is not uniform; it is greatest at their centre, and gradually decreases as we move towards the circumference of the sphere, where it is absolute zero. The law of this temperature is as follows: If R be the radius of the sphere, and r the distance of the point considered from the centre, the absolute temperature will be proportional to $R^2 - r^2$. Further, I shall suppose that in this world all bodies have the same coefficient of dilatation, so that the linear dilatation of any body is proportional to its absolute temperature. Finally, I shall assume that a body transported from one point to another of different temperature is instantaneously in thermal equilibrium with its new environment. ... If they construct a geometry, it will not be like ours, which is the study of the movements of our invariable solids; it will be the study of the changes of position which they will have thus distinguished, and will be 'non-Euclidean displacements,' and this will be non-Euclidean geometry. So that beings like ourselves, educated in such a world, will not have the same geometry as ours."

To his credit, M.C. Escher admitted that he did not understand a word of this, but he did finally see a logical way to represent infinity, with help from Coxeter, to keep diminishing regular shapes to be infinitesimally small. Escher wrote, "A diminution in the size of the figures ... from within outwards, leads to more satisfying results. The limit is no longer a point, but a line which borders the whole complex and gives it a logical boundary."

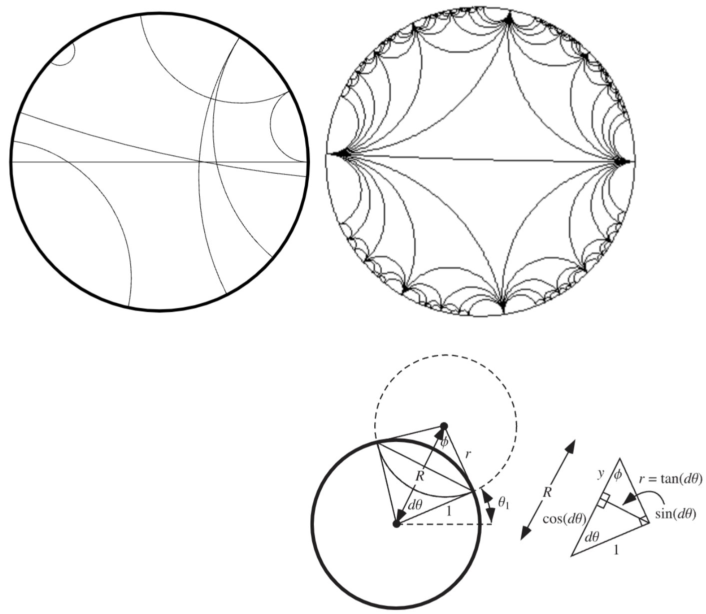


Figure 4: The Poincaré disk is a model for hyperbolic geometry in which a line is represented as an arc of a circle whose ends are perpendicular to the disk's boundary (and diameters are also permitted). Two arcs which do not meet correspond to parallel rays, arcs which meet orthogonally correspond to perpendicular lines, and arcs which meet on the boundary are a pair of limits rays. Credit: Wolfram Mathworld

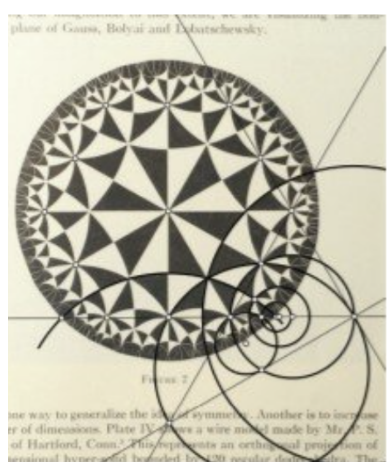


Figure 5: Computer-enhanced image of Escher's sketches on Coxeter's figure of a Poincare disk.

This work led to Escher's "Circle Limit" series and "Square Limit"; he called his method of construction "Coxetering":

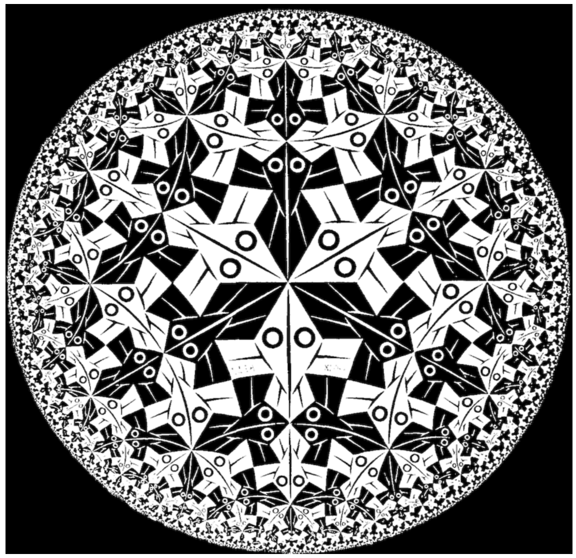


Figure 6: M.C. Escher, "Circle Limit I" (1958)

Figure 7: M.C. Escher, "Square Limit" (1964)

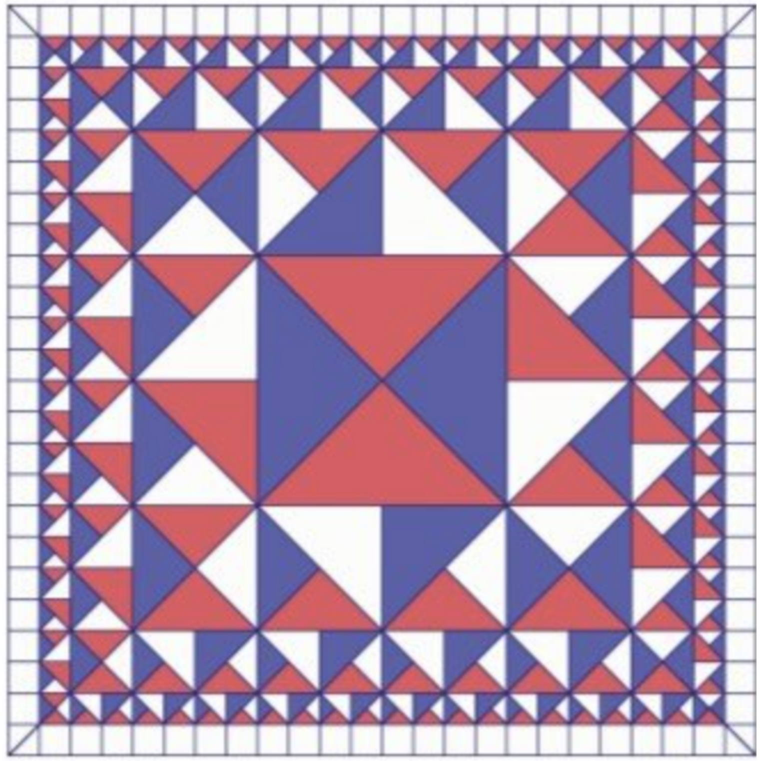


Figure 8: M.C. Escher's grid, used to construct "Square Limit" that he shared with Coxeter

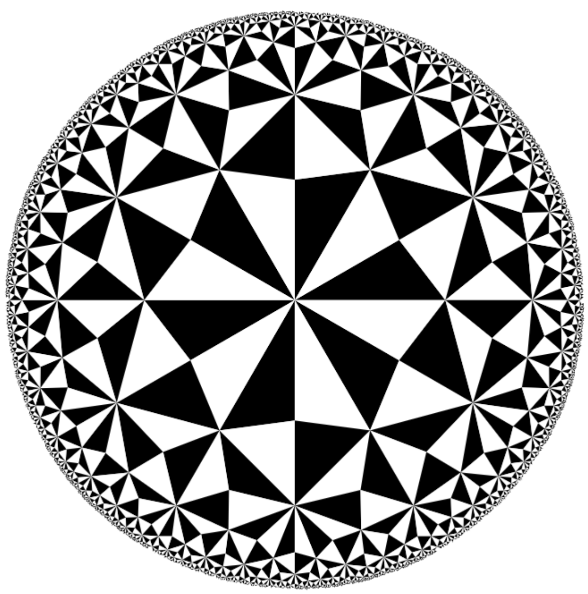


Figure 9: A hyperbolic tessellation similar to M. C. Escher's "Circle Limit IV (Heaven and Hell)." Credit: Wolfram Mathworld, Trott 1999.

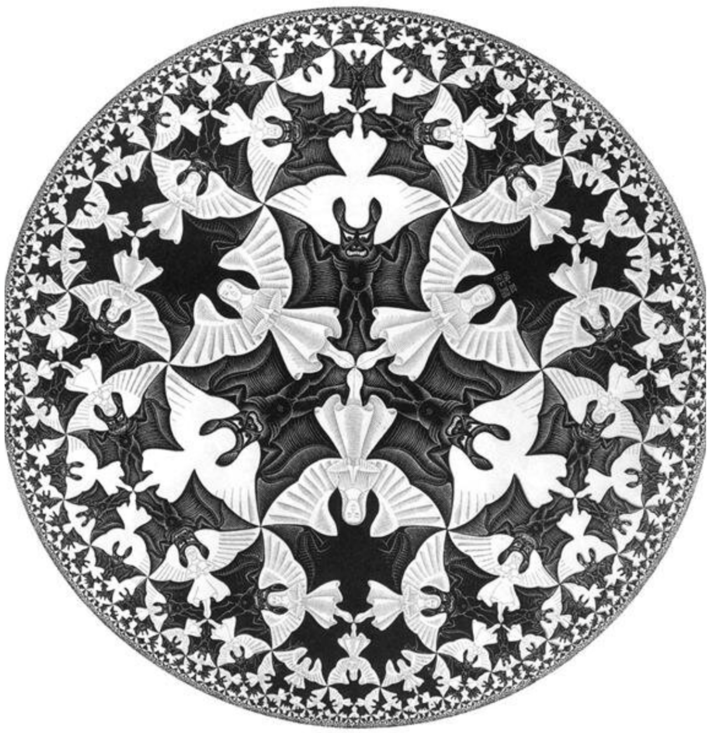


Figure 10: M.C. Escher, “Circle Limit IV (Heaven and Hell)” (1960)