What is proportion?

Art of Mathematics, Summer 2023

June 6, 2023

First, from last time: the Texas tilings on a street in North Texas!

1 Introduction

Today we'll talk about ratios and proportion.

Ratios give us a relationship between the sizes (or magnitudes) of two quantities. For example, in making a lemonade recipe, you may have seen ratios written in one of the following ways:

$$1 \text{ part lemon juice to } 3 \text{ parts water, or}$$

$$1{:}3 \text{ lemon juice to water, or}$$

$$\frac{1 \text{ part lemon juice}}{3 \text{ parts water}}.$$

This means that no matter what units we use to measure lemon juice or water (gallons, quarts, milliliters, etc.), the ratio between them should always stay constant: 1/3. In other words, there should always be 3x more water than lemon juice.

Proportions are closely related to ratios: a proportion is created when two ratios are set equal to each other. In other words, proportions state that two ratios are equivalent.

Going back to the lemonade example, suppose I make one batch of lemonade with 1 gallon of lemon juice and 3 gallons of water. Then suppose I double the recipe for a different batch of lemonade: 2 gallons of lemon juice and 6 gallons of water. Even though the second batch uses more ingredients, the ratio of those ingredients is the same in both–which, for us, means the two batches should taste the same! The proportion that states this in the language of math would be: $1/3 = 2/6$.

2 The Golden Ratio

We'll start by talking about the Fibonacci sequence, described in the 13th century by the Italian mathematician Leonardo de Pisa (also known as Fibonacci):

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

What do you notice about this pattern? How would you find the next number in the sequence?

As you continue to write down numbers in the Fibonacci sequence, something very interesting happens in the ratio of terms. Specifically, the ratio approaches 1.61803398874... Let's call it φ. Early Greek artists and philosophers used this number φ to design Greek architecture.

The Greeks believed that the perfect relationship between the width and height of buildings is:

$$\varphi = \frac{\text{width}}{\text{height}}.$$

If we let w stand for width, and h for height, then the Greeks expressed this perfect relationship as the "Golden proportion"

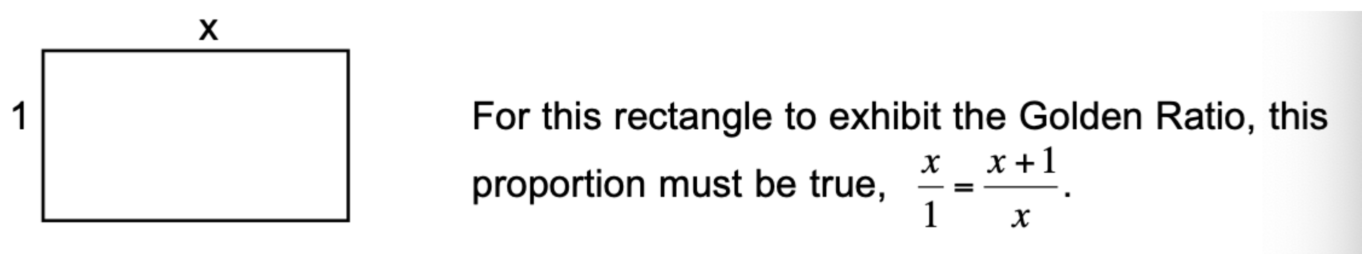
$$\frac{w}{h} = \frac{w+h}{w}.$$

In other words, the ratio of the *width* of a building to its *height*, should be the same as the ratio of its *width plus its height* to its *width*. We can see an example of this in the Parthenon:

Figure 1: The Parthenon in Athens, Greece, a temple dedicated to the goddess Athena built in 5th century BC.

Let's briefly investigate how this φ from the Greeks is the same as the value approached by the ratio of Fibonacci numbers.

Pretend we have a Greek building with a height of 1 unit:



Let's solve this proportion for what the width x would need to be:

$$x^2 = x + 1$$
$$x^2 - x - 1 = 0.$$

Using the quadratic formula, we get two solutions for $x = \frac{1 \pm \sqrt{5}}{2}$. Since $\frac{1-\sqrt{5}}{2}$ is a negative number, it doesn't make sense for it to be the width of a building. So the solution for the width x must be $\frac{1+\sqrt{5}}{2}$, and indeed, this is the golden ratio $\varphi \approx 1.618033989$!

Now, in the Fibonacci sequence, we can divide numbers to see what the ratios approach:

$$\frac{1}{1} = 1$$
$$\frac{2}{1} = 2$$
$$\frac{3}{2} = 1.5$$
$$\frac{5}{3} = 1.\overline{6}$$
$$\frac{8}{5} = 1.6$$
$$\frac{13}{8} = 1.625$$
$$\frac{21}{13} = 1.\overline{615384}$$
$$\frac{34}{21} = 1.\overline{619047}$$
$$\frac{55}{34} = 1.6\overline{1764705882352941}$$

3 Visualizing the Fibonacci Sequence

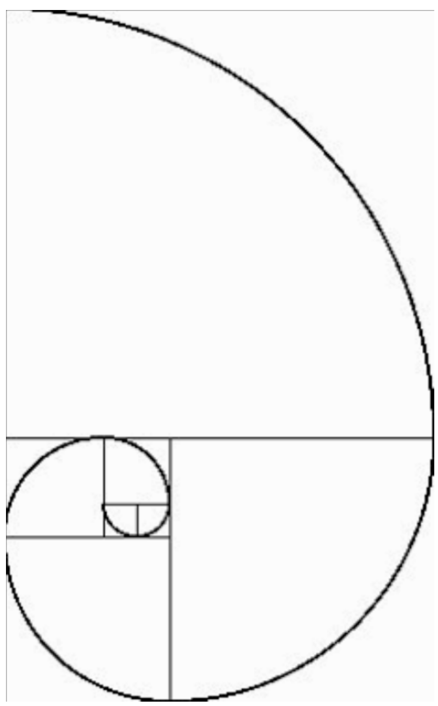


Figure 2: The Fibonacci spiral (image courtesy of Leslie Lewis, ldlewis.com)

We can view the Fibonacci sequence as a spiral by drawing successive Fibonacci sized squares.

Start with a 1x1 square in the center of the paper. Now, each new square you draw is going to be placed in a counter-clockwise direction to the previous square, all rotating around this initial one.

Beside it, draw another 1x1 square.

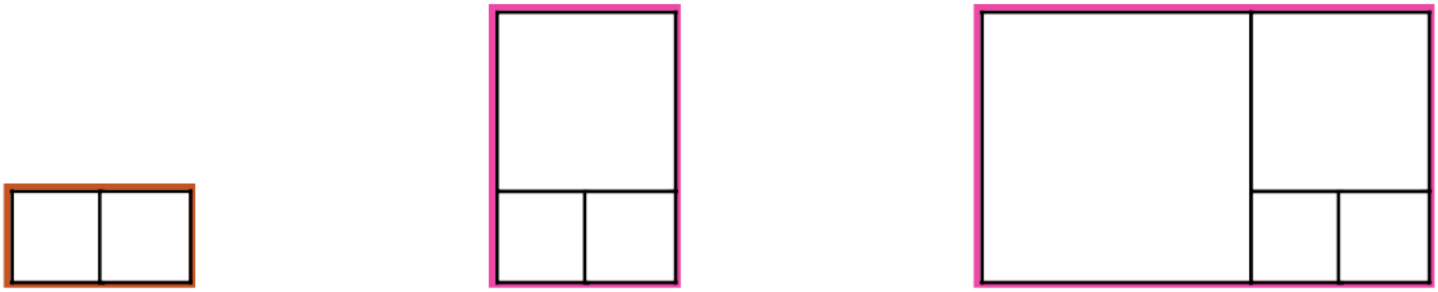


Figure 3: Image courtesy of Leslie Lewis, ldlewis.com

Since $1 + 1 = 2$, draw a 2x2 square above the previous two squares.

Then the next square, drawn to the left of the previous ones, will be 3x3, and so on.

We can fit all of the squares at any step inside of a rectangle. Those rectangles have a height to width ratio approaching φ.

To get the spiral, we draw a quarter-circle inside each square, with the center being the top corner of our first square:

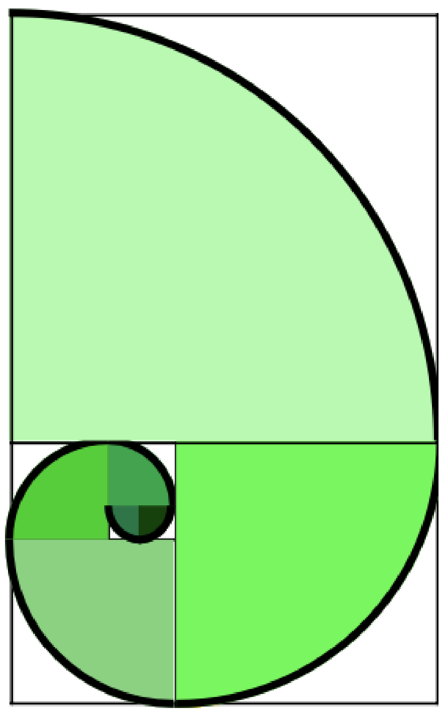


Figure 4: Image courtesy of Leslie Lewis, ldlewis.com

4 Project Ideas

Use the Golden Ratio φ in a sketch of something from nature, architecture, or your own imagination that exhibits φ.

Figure 5: Image courtesy of Chris Chadwick, TotemLearning.com